Numerical methods for collapse analysis of RC framed structures with physical and geometrical nonlinearity

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3. Energetic consistency

Numerical methods for the integration of the equations of motion should always preserve the physical meaning of the solution.

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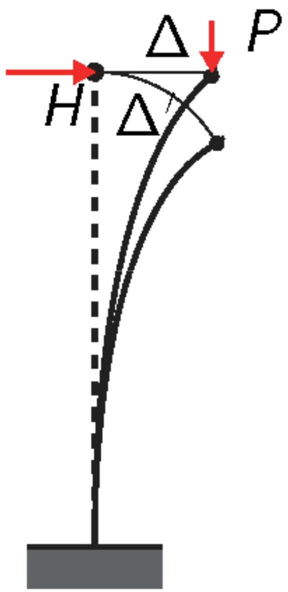
- Use an **extended representation for node displacements in space**, to enhance numerical accuracy and computational performances
- Adapt the **corotational formulation** of a force-based beam element to this new extended set of coordinates
- Develop a finite element code using this theory, implementing stable solution procedures based on concepts of **differential geometry**

Geometrical nonlinearity



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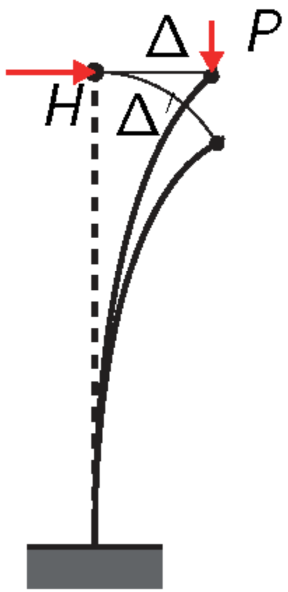

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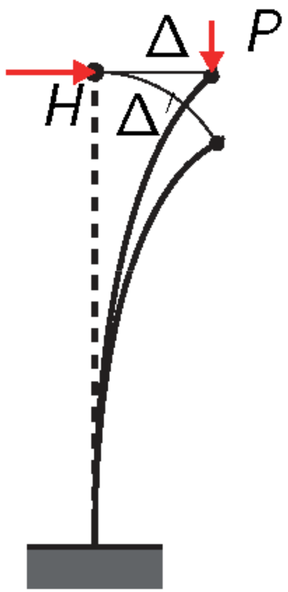

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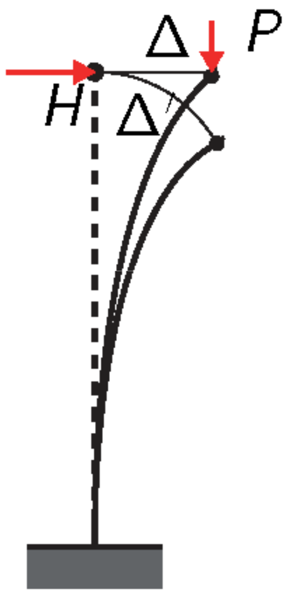

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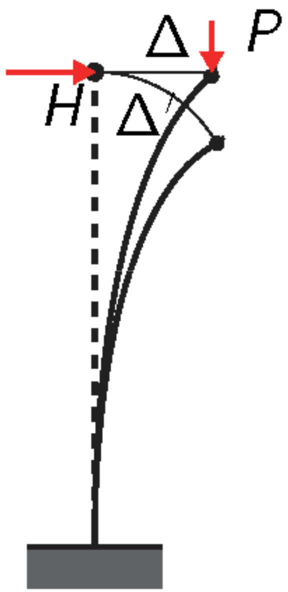

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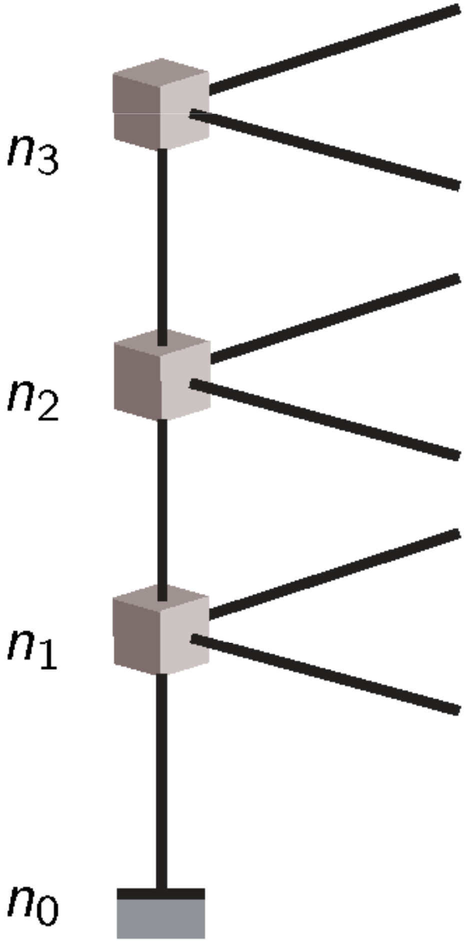
Spatial rotations. The application of two consecutive rotations cannot be treated as an algebraic addition. In fact, **rotations have a non-vectorial nature** that should be considered for their composition.

Structural system



Framed structure

Mechanical system composed by rigid bodies (**nodes**) connected by elastic/inelastic links (**elements**).

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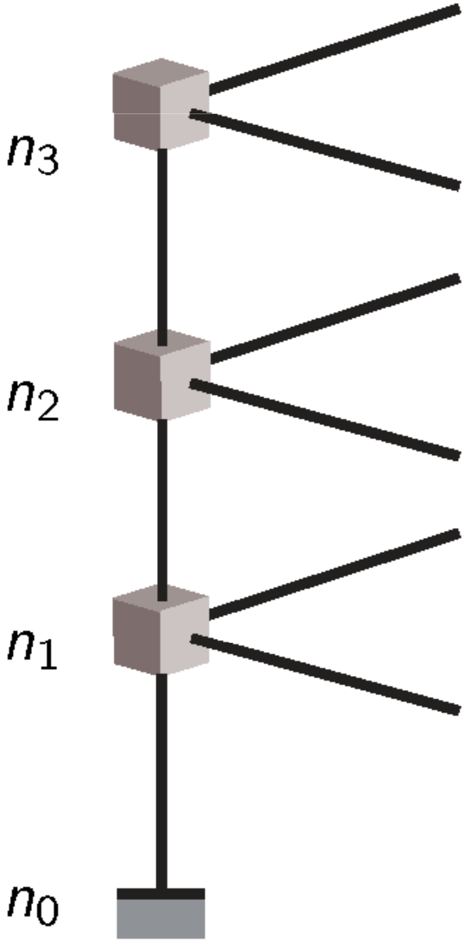

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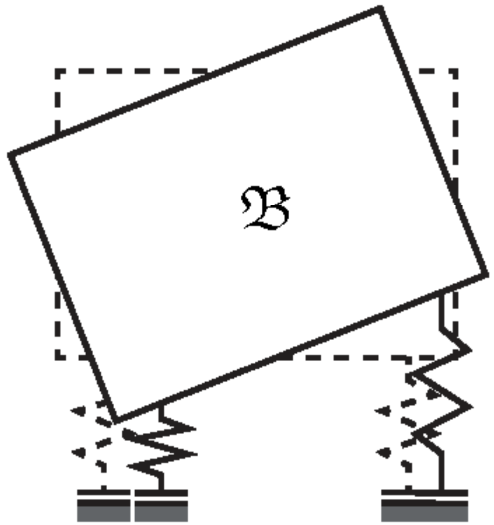
A definition of such a system requires an *unique* description for the configuration of a rigid body $\mathfrak{B}$.



Configuration space of a node

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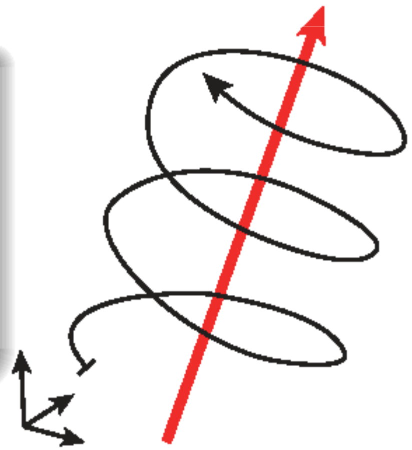

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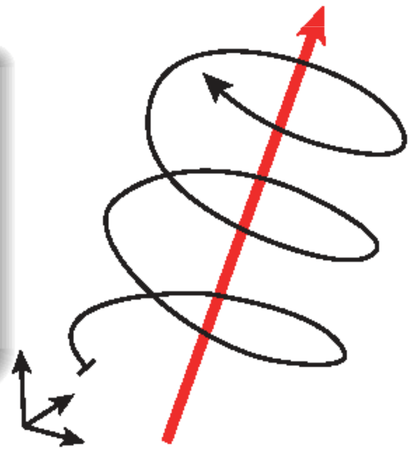

The most classical way for representing numerically a screw motion is by using a homogeneous 4x4 matrix **Rbm**:

$$\mathbf{Rbm} = \begin{pmatrix} \mathbf{R} & \mathbf{q}_v \\ \mathbf{0}^T & 1 \end{pmatrix} \tag{1}$$

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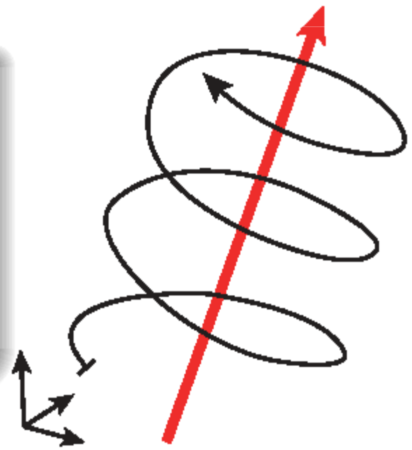

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Where $\mathbf{q}_v$ is the translation vector and $\mathbf{R}$ is the rotation matrix, which is an element of the orthogonal group $SO(3)$.

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The rotation matrix is a 3x3 *orthogonal matrix* with $\det(\mathbf{R}) = 1$. Due to its properties, it can also be represented by a more concise set of parameters.

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Each of these can be *converted* into a rotation matrix, by means of a proper mapping function $\mathbf{R}_{(\ldots)}(\mathbf{q})$, which is not always **bijective**.

Rotation matrix - Euler angles (1)

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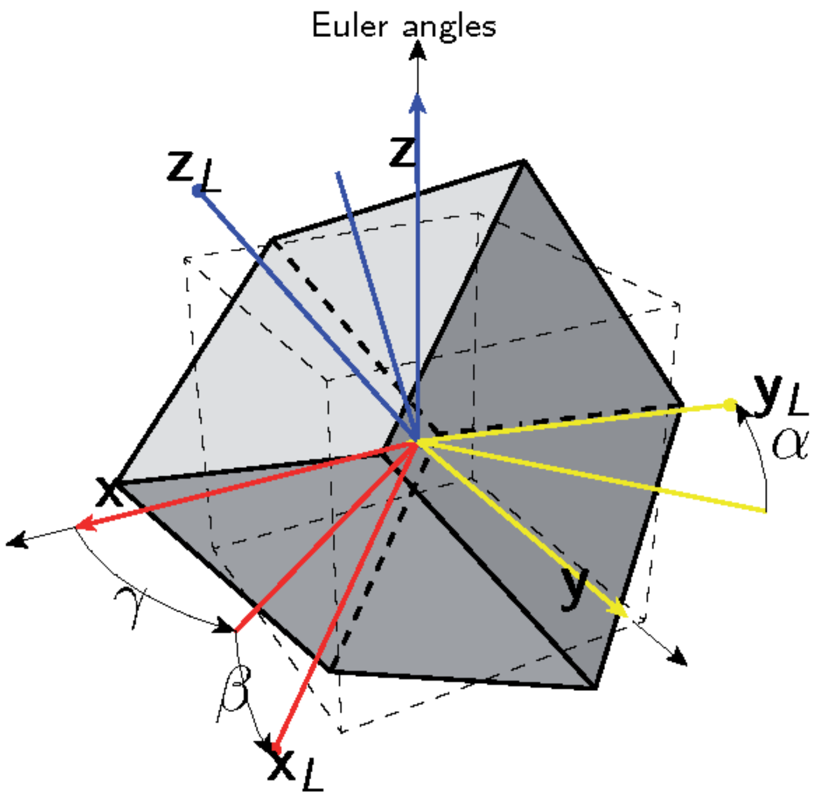

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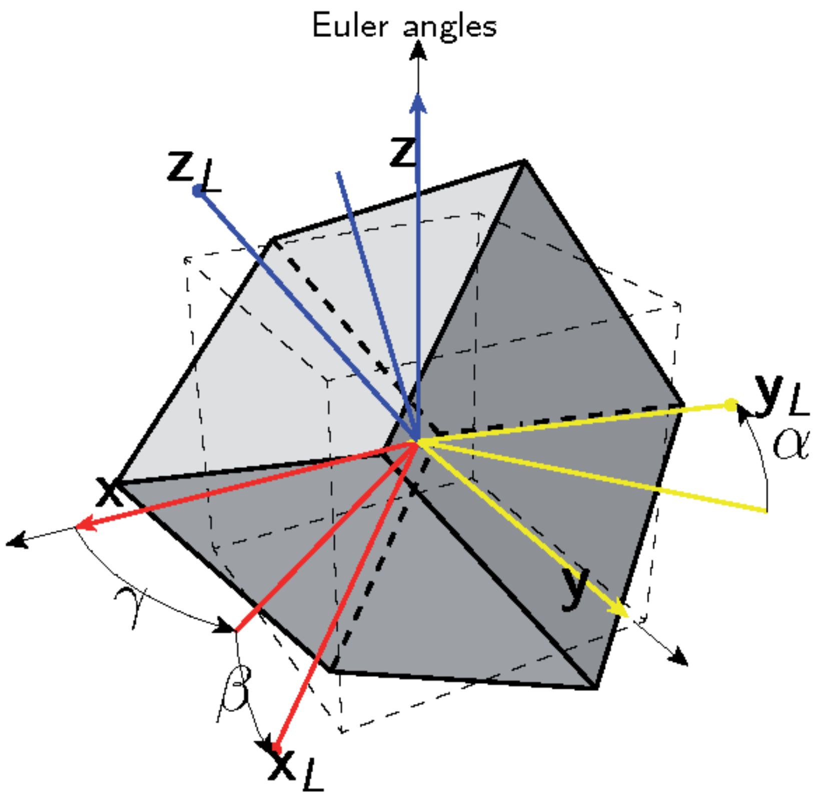

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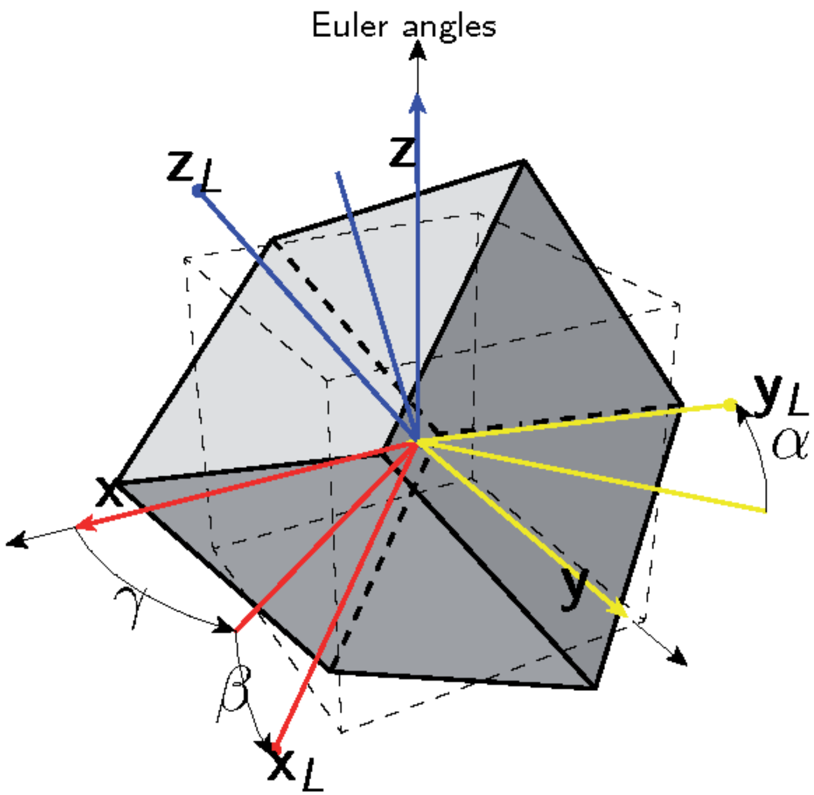

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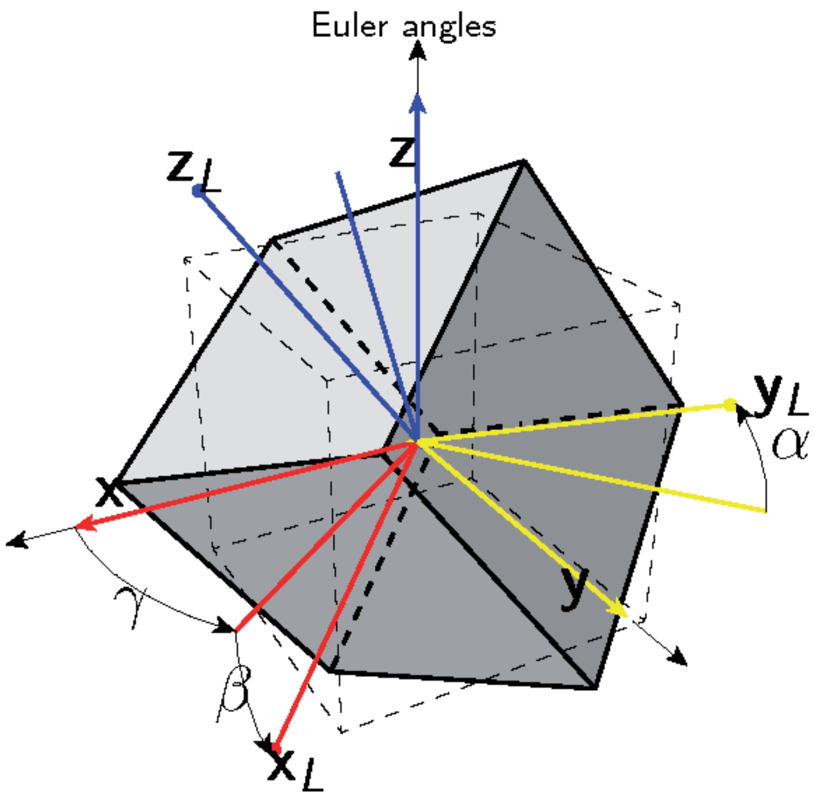

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So that the total rotation matrix $\mathbf{R}(\alpha, \beta, \gamma)$ is given by:

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It can be shown that for a critical angle $\beta = \pm\pi/2$, the same rotation can be obtained by different sets of the other two parameters:

$$\mathbf{R}_{\mathbf{eul}}(0, -\pi/2, 1) = \begin{pmatrix} 0 & 0 & -1 \\ 0.8414 & 0.5403 & 0 \\ 0.5403 & -0.8414 & 0 \end{pmatrix}$$

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This means it is not possible to go back to the full rotation matrix in a unique way, which causes **singularities**.

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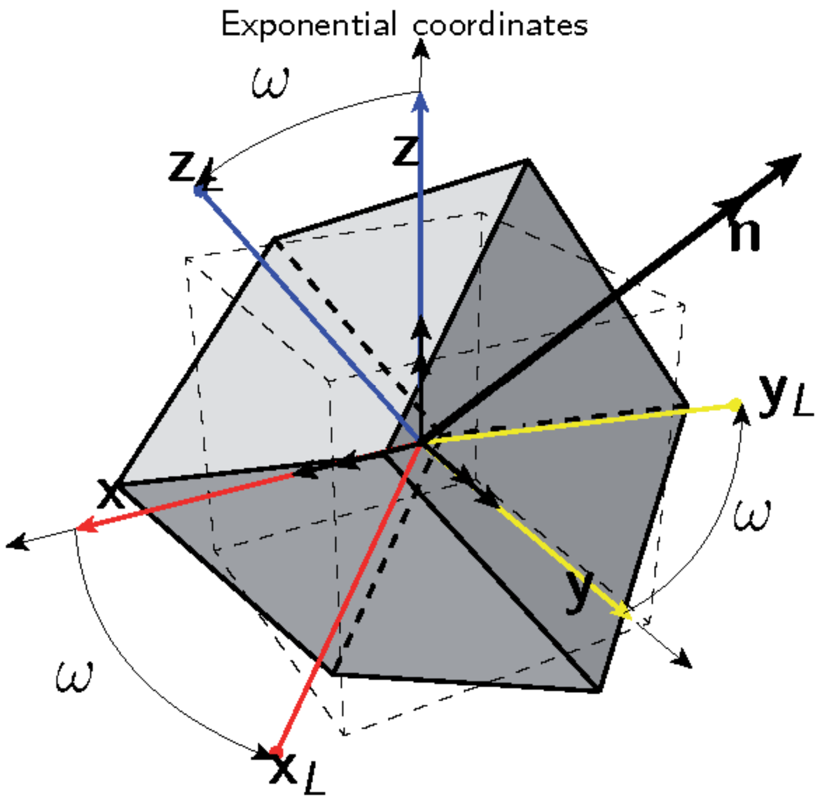

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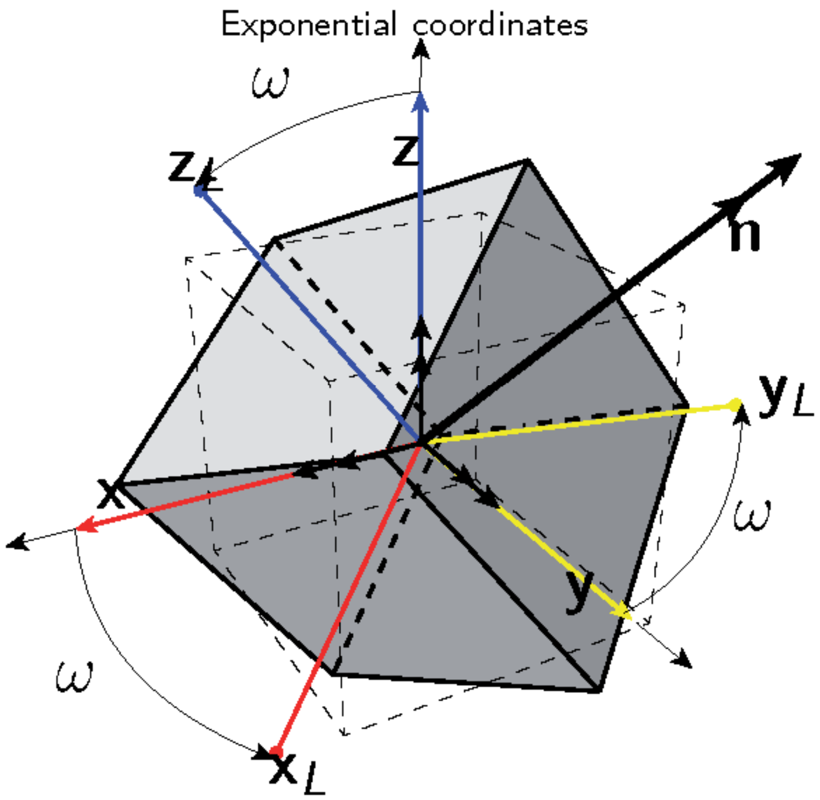

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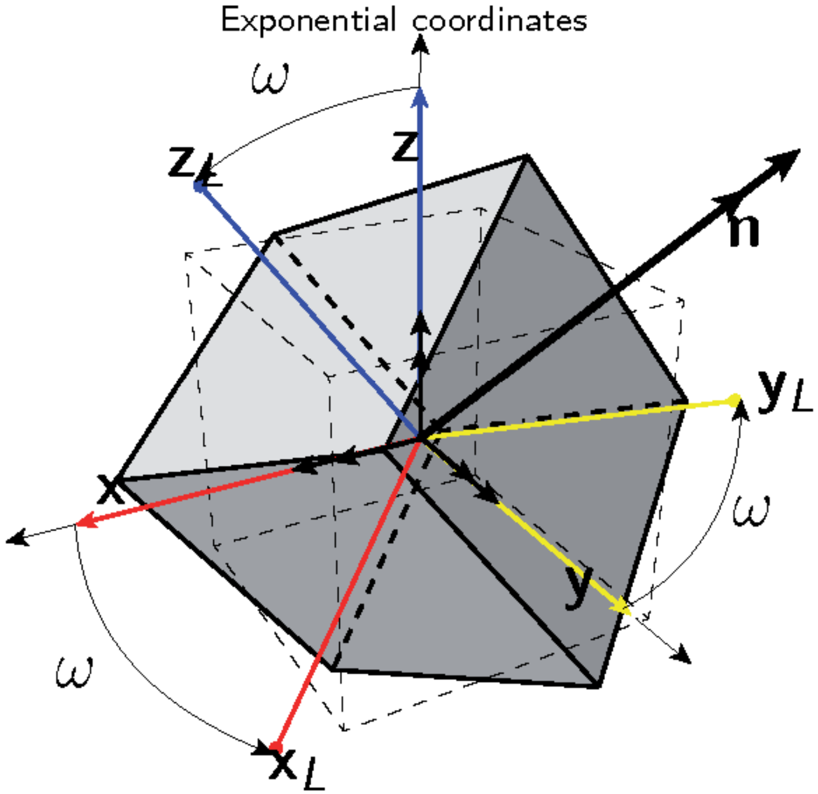

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It is important to point out that the total number of parameters is equal to three, since the third component of the unit vector can be obtained as the cross product of the other two.

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It means that, in case $\omega = 0$, any value of $\mathbf{n}$ produces $\mathbf{R_{exp}} = I_3$. Therefore, a loss in accuracy could be expected in **quasi-translational** motion.

Rotation matrix - Unit quaternions(1)

Unit quaternions are elements of the group *Spin*(3), which is the *double cover* of $SO(3)$, and can be defined as

$$\mathbf{q} = \mathbf{Q}(q_0, q_1, q_2, q_3) = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \tag{7}$$

with $\|\mathbf{q}\| = 1$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$.

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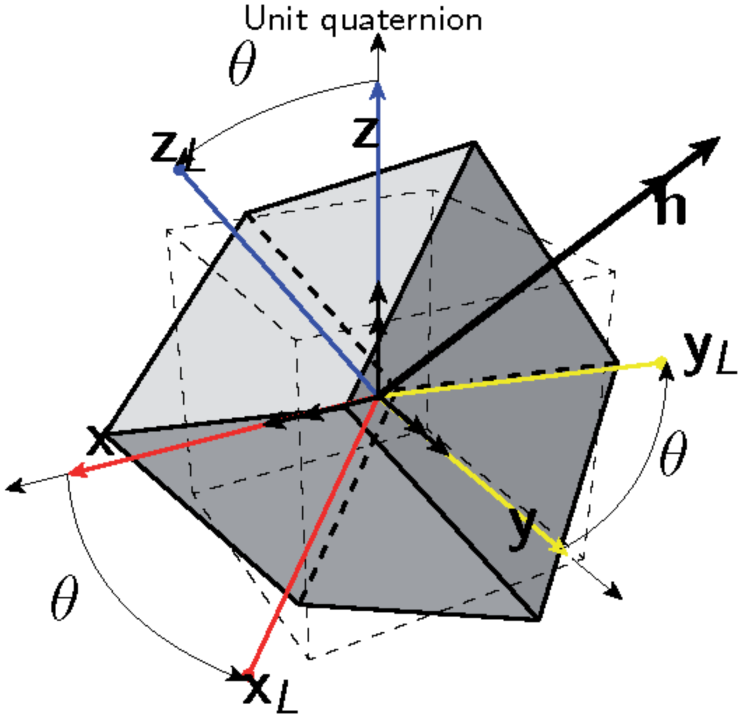
Unit quaternions have a geometrical interpretation given by:



$$\mathbf{q} = \cos\frac{\theta}{2} + \mathbf{h}\sin\frac{\theta}{2} \tag{8}$$

where θ is the angle of rotation and $\mathbf{h}$ is the unit vector of the rotation axis.

Rotation matrix - Unit quaternions(2)

A rotation matrix can be obtained from a quaternion **q** as follows:

$$\mathbf{R}_{\mathbf{quat}}(q_0, q_1, q_2, q_3) = \begin{pmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_1 q_2 + q_0 q_3) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_0 q_3) & 1 - 2(q_1^2 + q_3^2) & 2(q_2 q_3 + q_0 q_1) \\ 2(q_1 q_3 + q_0 q_2) & 2(q_2 q_3 - q_0 q_1) & 1 - 2(q_1^2 + q_2^2) \end{pmatrix} \quad (9)$$

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Two opposite quaternions represent the same rotation:

$$\mathbf{R}_{\mathbf{quat}}(0.5403, 0, 0, 0.8415) = \begin{pmatrix} -0.4161 & 0.9093 & 0 \\ -0.9093 & -0.4161 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$|\mathbf{q}|$ **is in unique correspondence with the rotation matrix**, and so it constitutes the best representation method for rotations.

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Both solutions are considered in the study, and used in different contexts.

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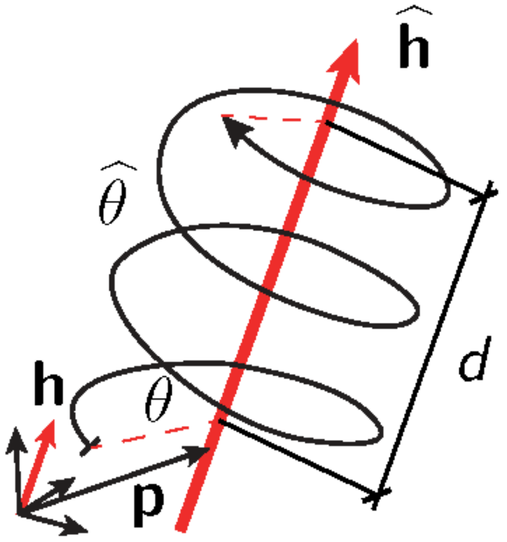

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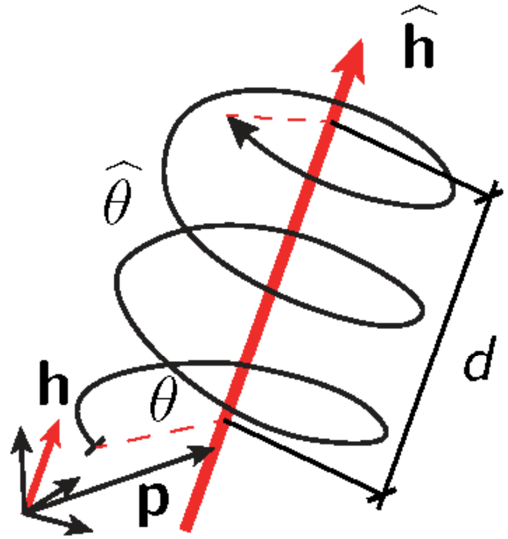

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Principle of transference

All the algebraic properties of quaternions hold for dual quaternions[3].

Case study - Description of the system

Nontrivial sample system for the study of rigid body dynamics.

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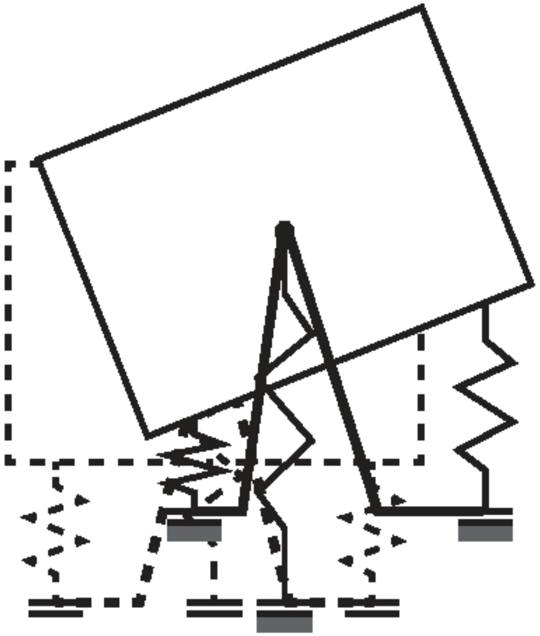

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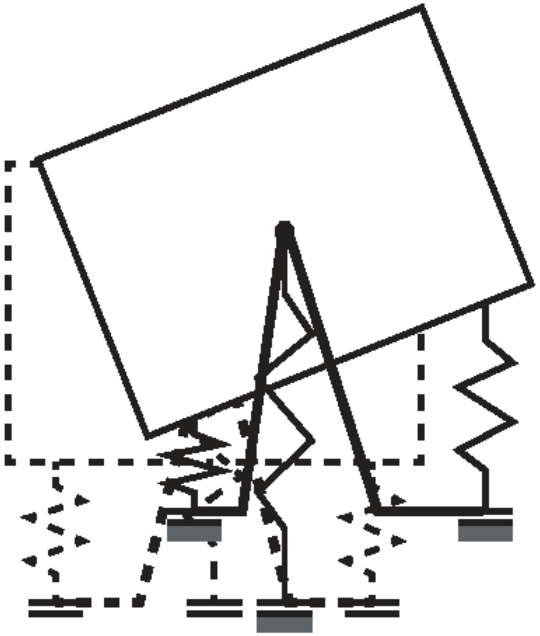

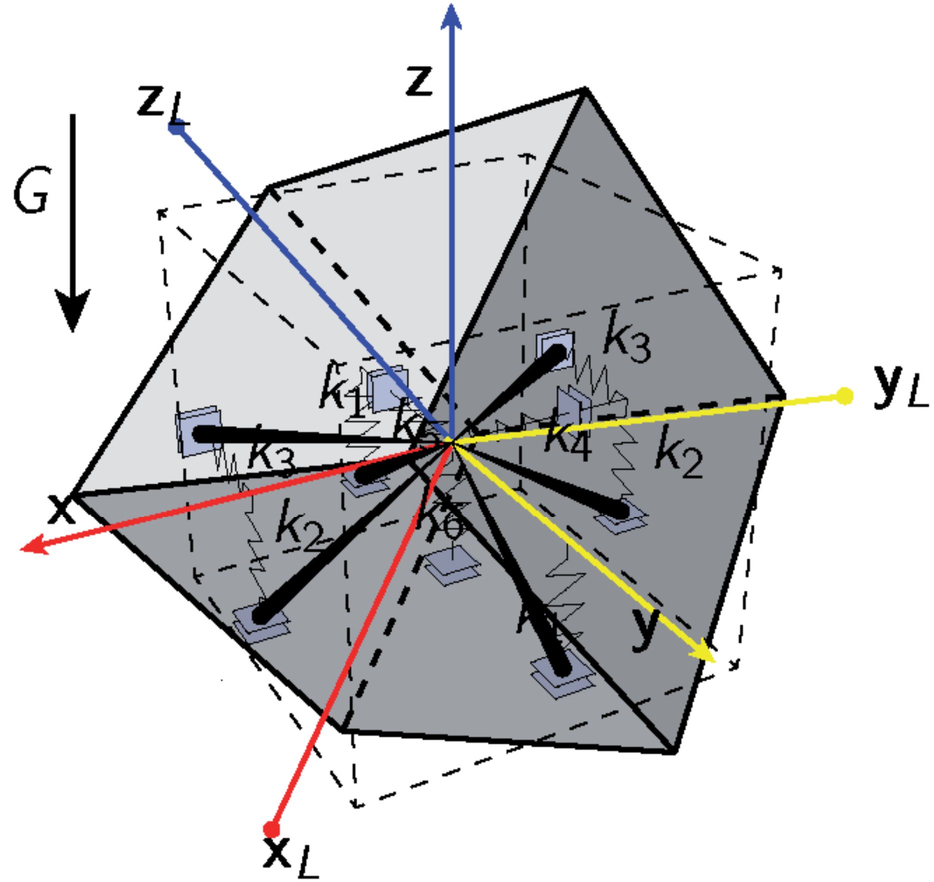

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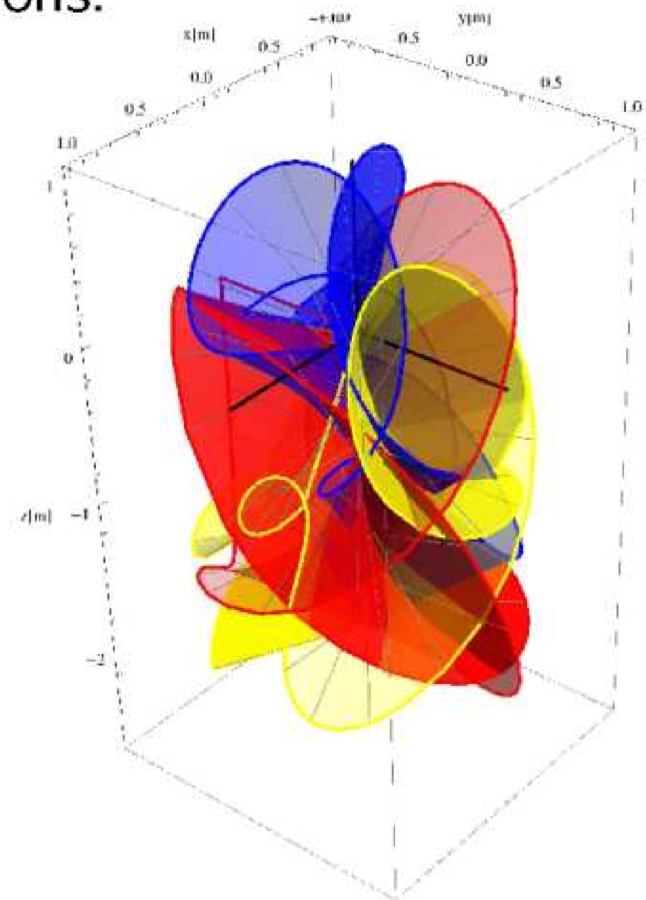
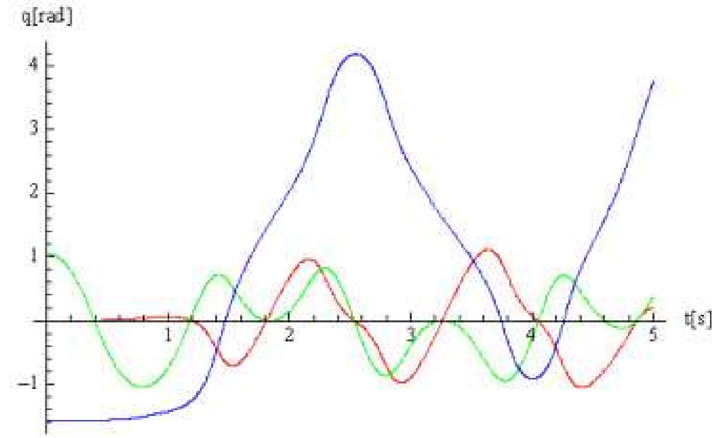
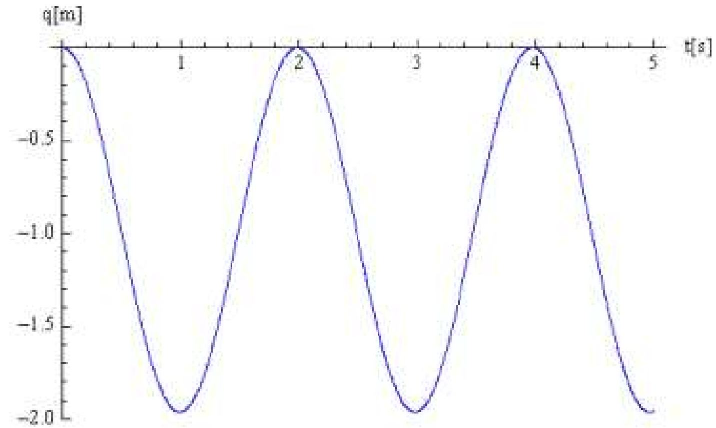


Initial conditions: $\{0, \pi/3, -\pi/2, 0, 0, 0\}$, gravity



Euler angles history



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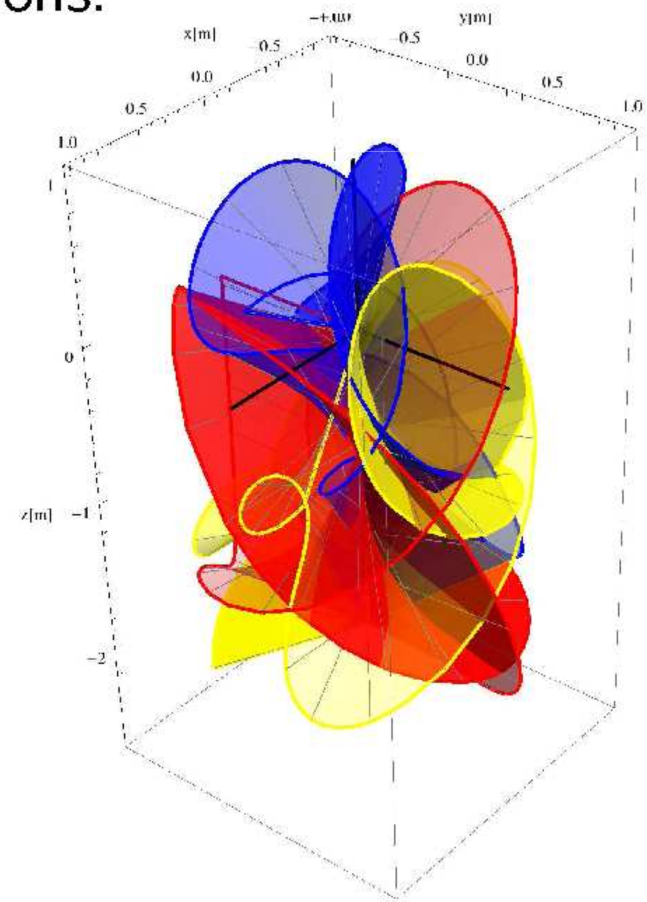

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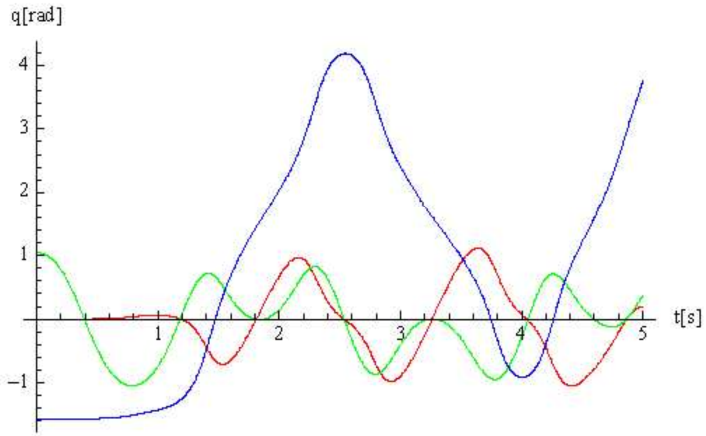

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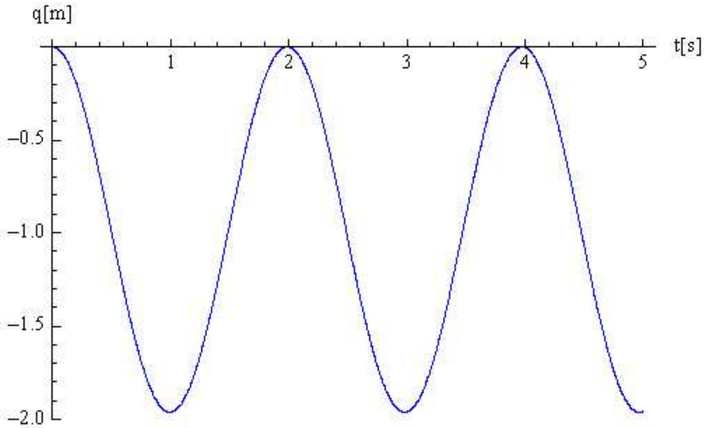

Translations history

Rotation angles are **coupled**. Rotation angles and translations are **uncoupled**.

Case study - Results of the implemented procedure

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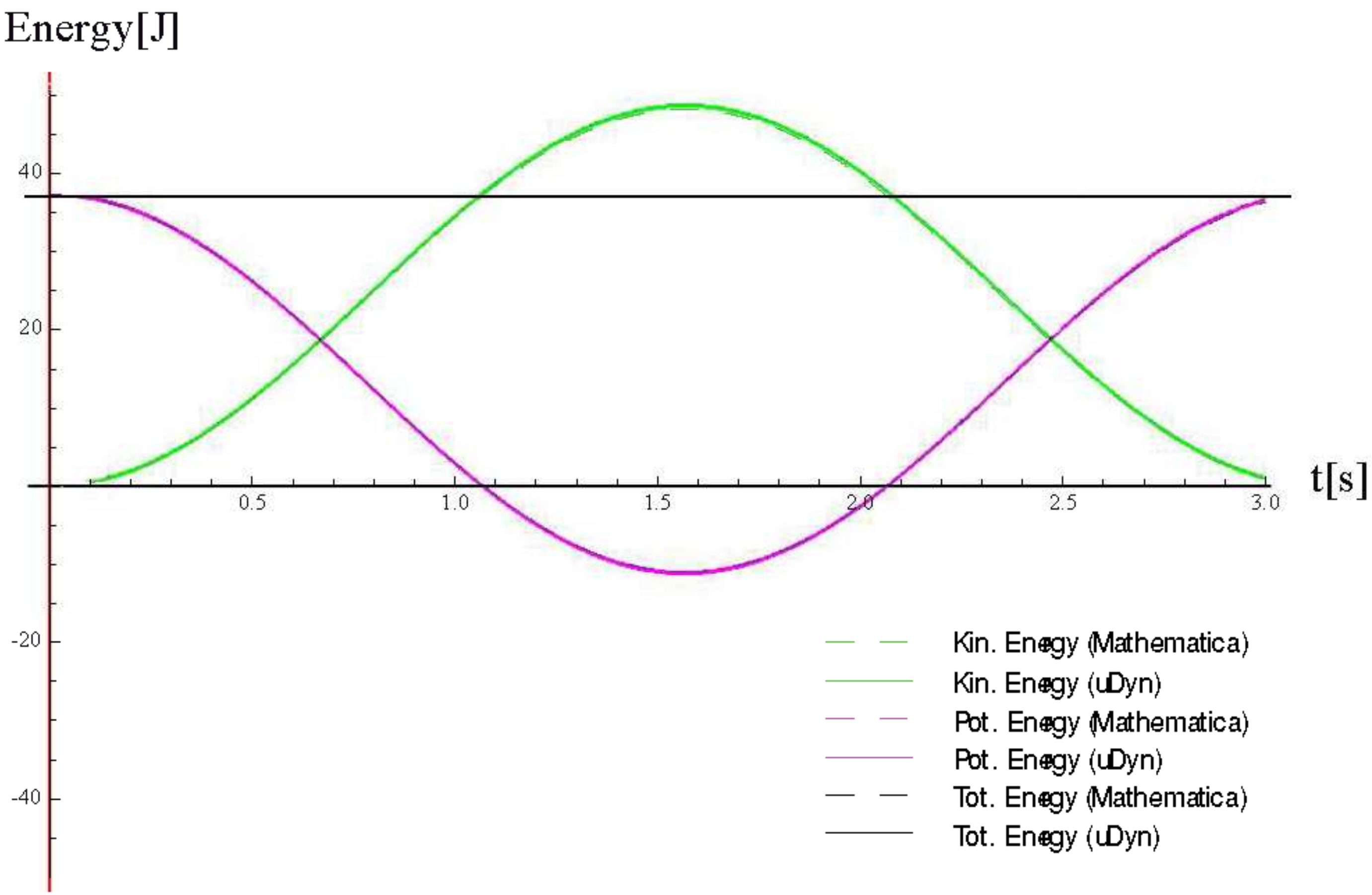

Plot of Energy profiles of the sample system

Enhanced corotational formulation

The following are considered:

- small strains along the element
- large displacements of the nodes

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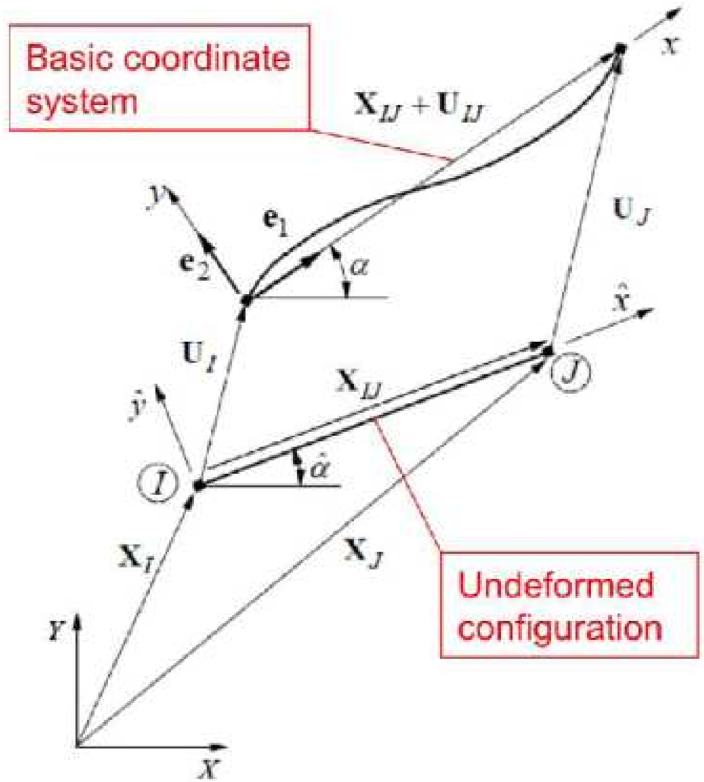

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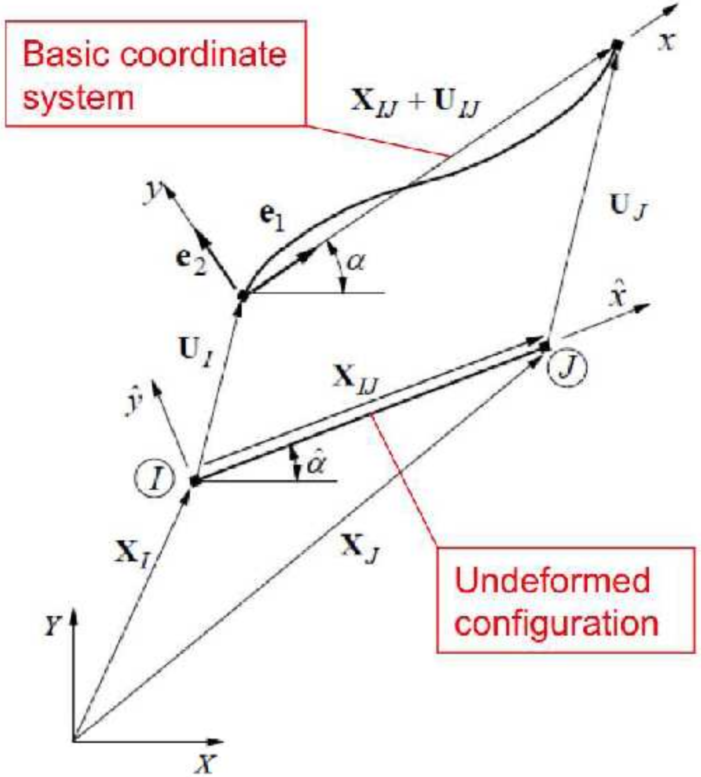
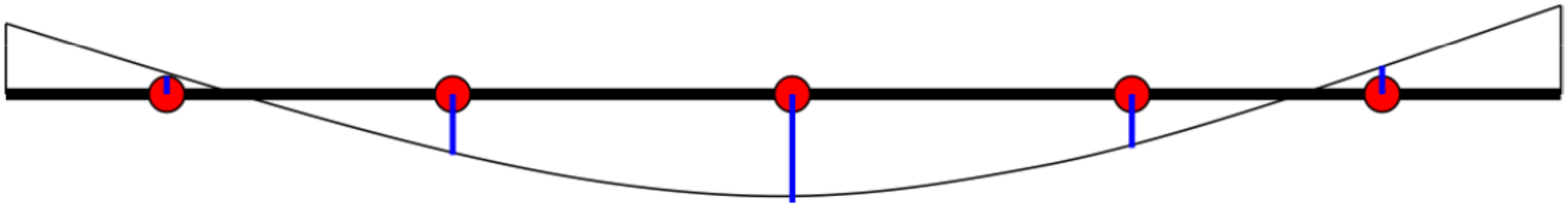
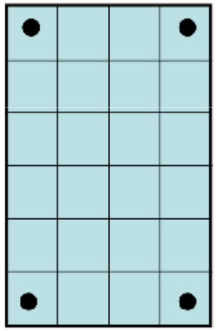


Large nodal displacements, expressed as **augmented nodal coordinates** are converted to the classical 6-DOFs so it can be used with traditionally-formulated beam models.
Therefore, a mapping function from enhanced coordinates to $\mathbb{R}^6$ is used at this scope.

Element integration

We use a force interpolation matrix $\mathbf{b}(\xi)$ to evaluate the **exact** forces at the fiber-discretized sections[1].





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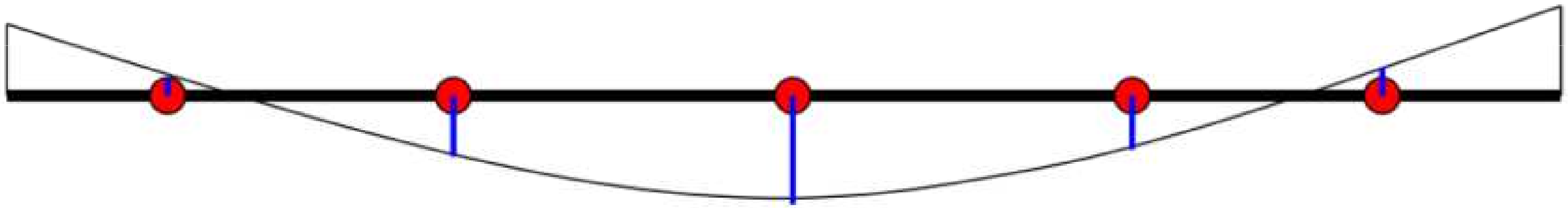
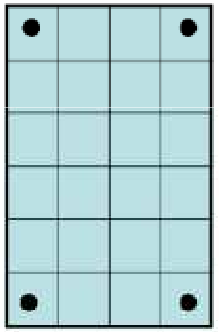
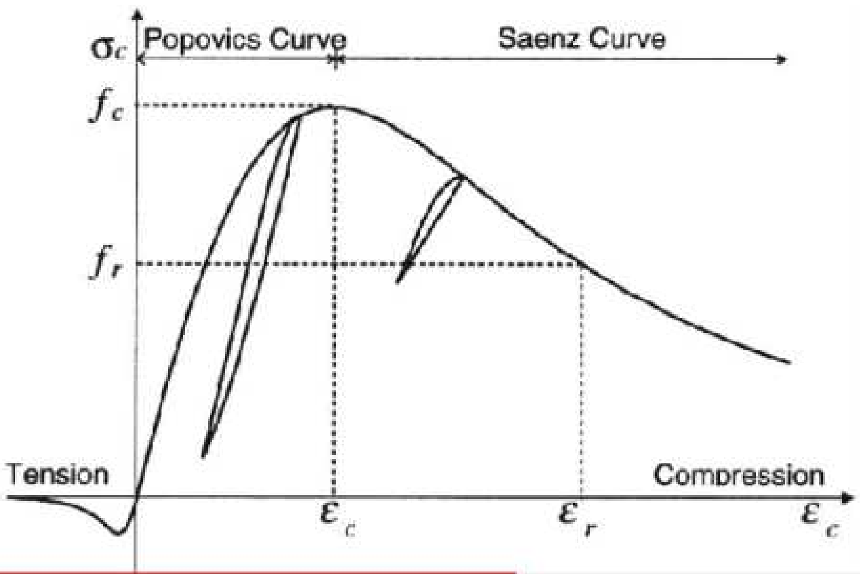
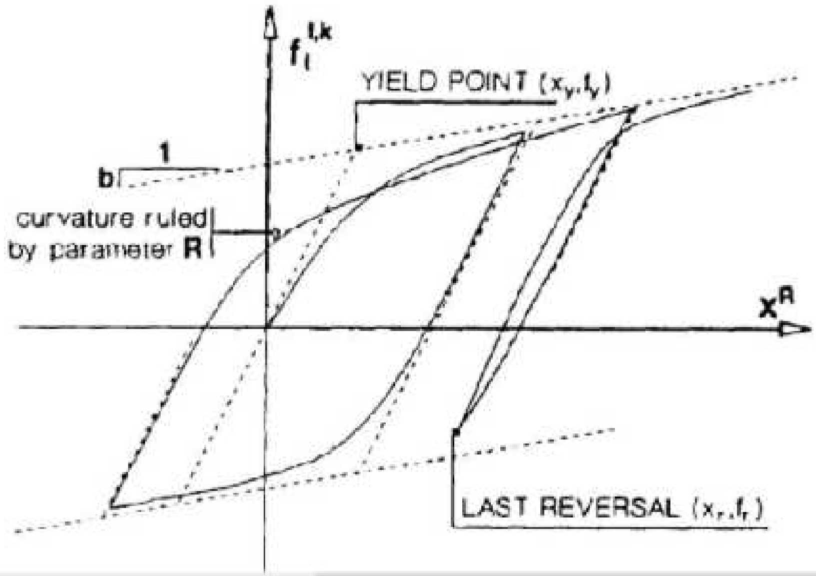




The following hysteretic constitutive models with cyclic behavior are considered for the axial behavior of the fibres:





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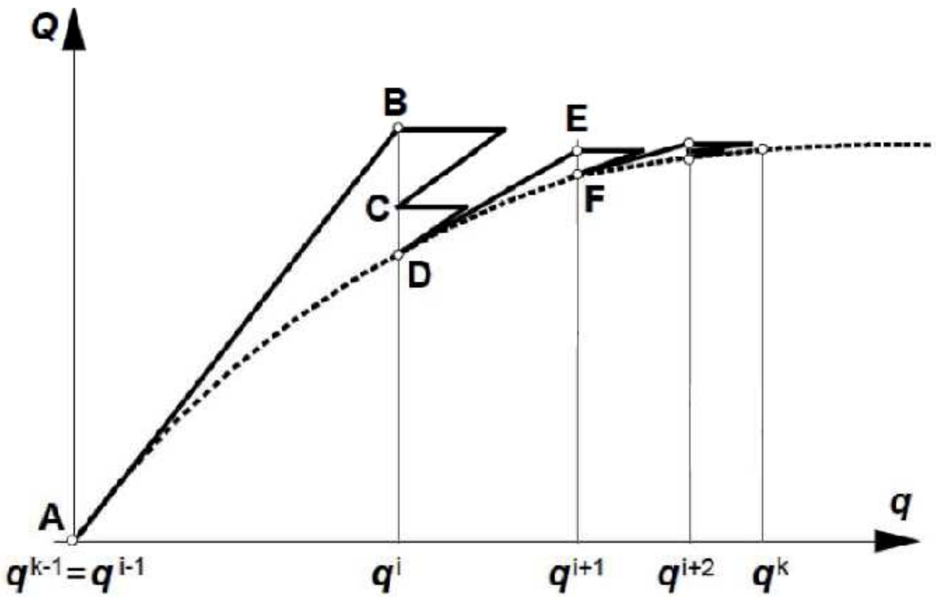

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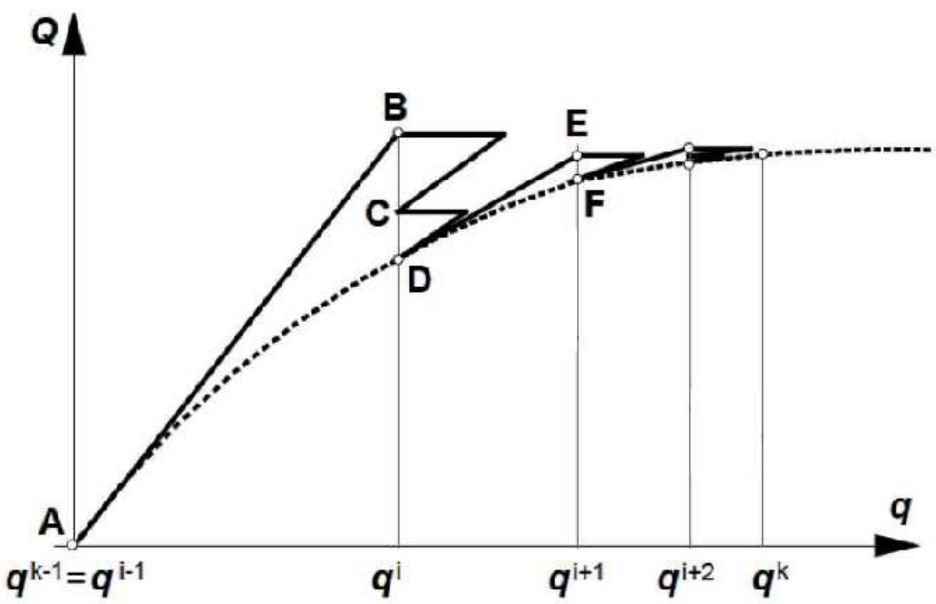
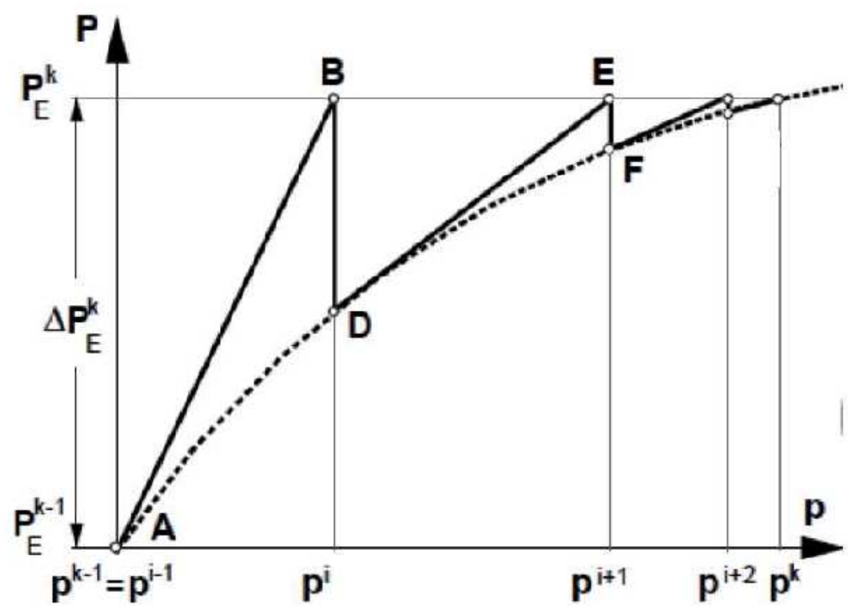
Structure iteration level

At the structure level, equilibrium is reached by solving for the equilibrium of the global system.





Considered aspects - 3. Energetic consistency

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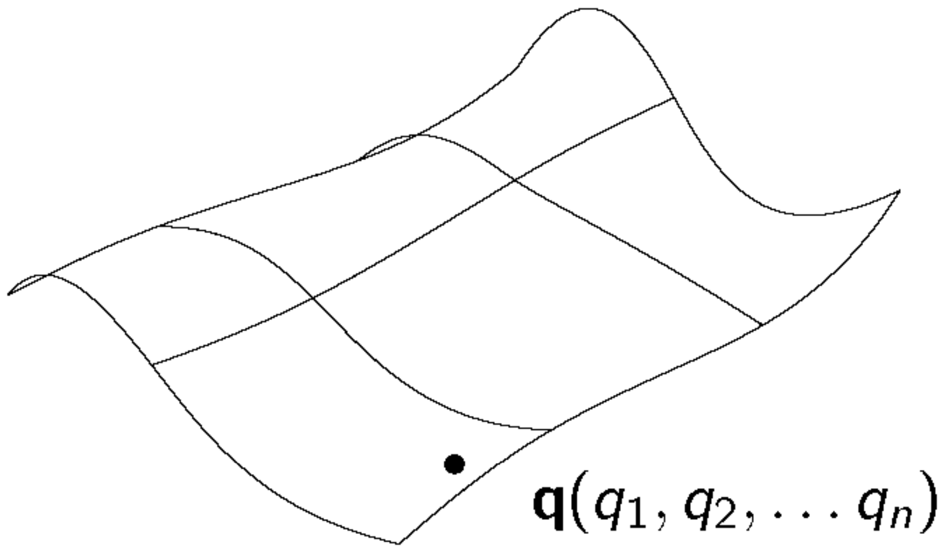

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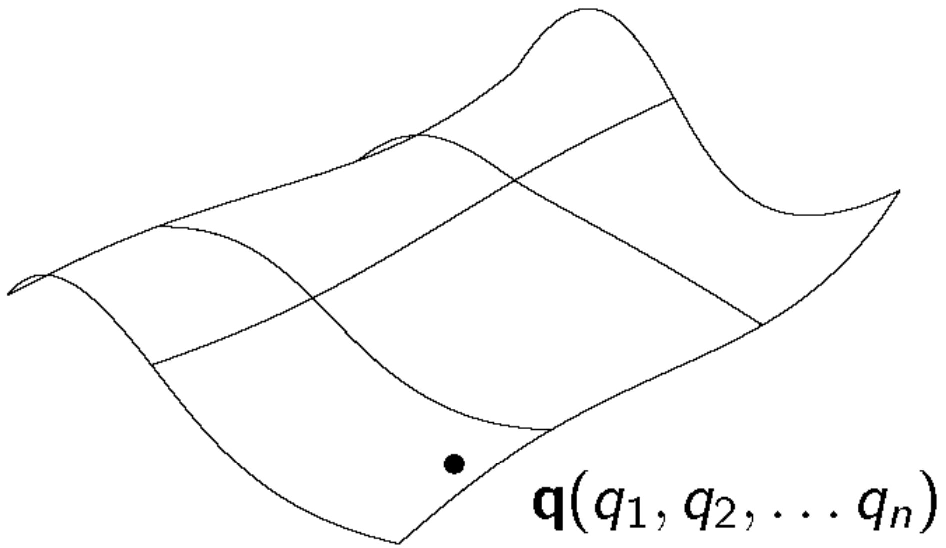

A stable computational method **preserves such structure** while performing numerical integration, and **conserves some quantities characterizing the mechanical problem** (*first integrals*).

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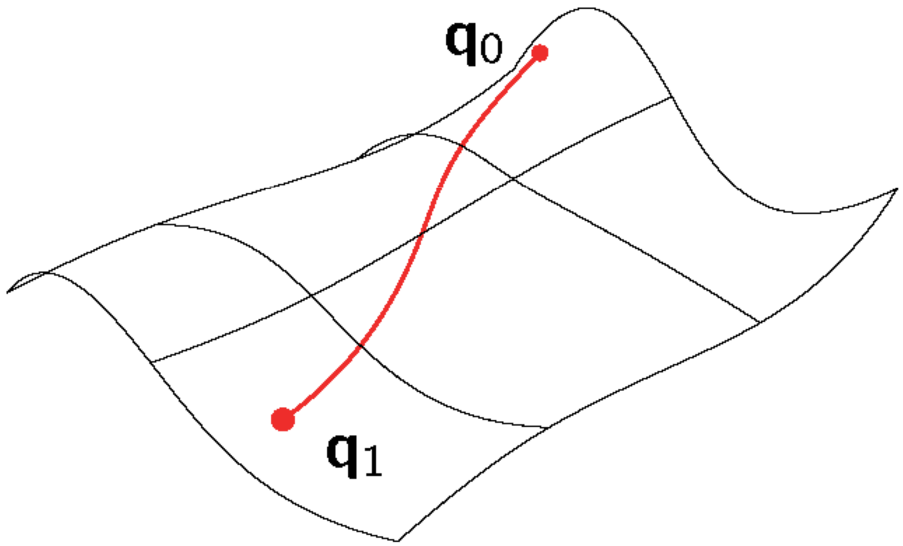

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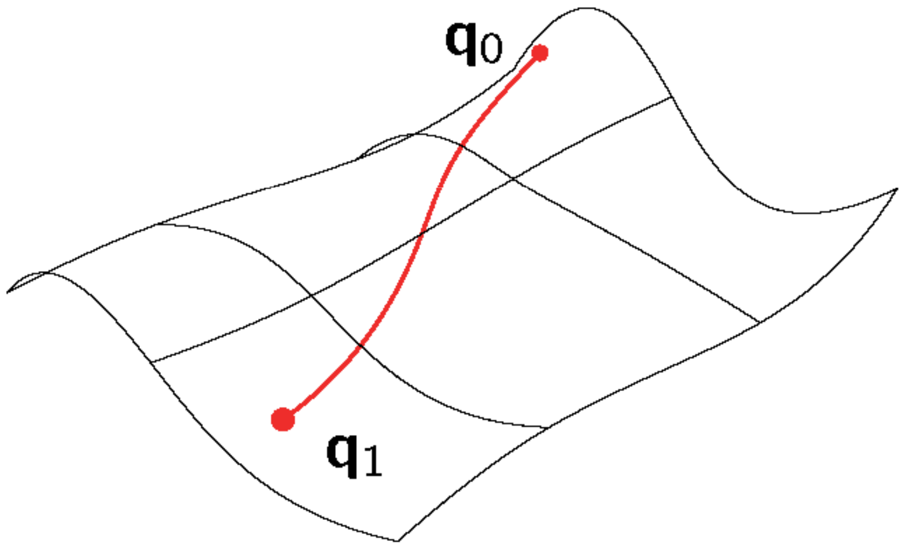

The system can be written in *Hamiltonian* form as a system of first order differential equations (*symplectic system*):

$$\begin{cases} \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \end{cases} \tag{13}$$

where the Hamiltonian H of the system is $H = T + V$.

Numerical solution and energetic consistency

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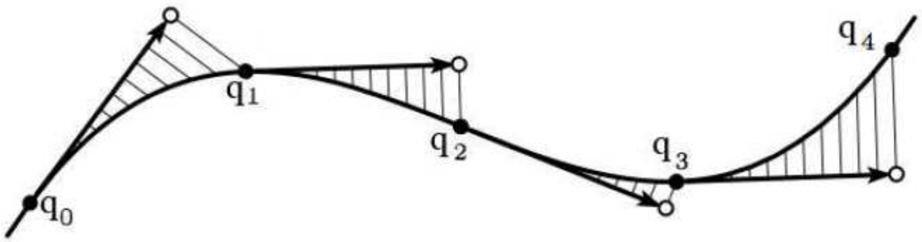
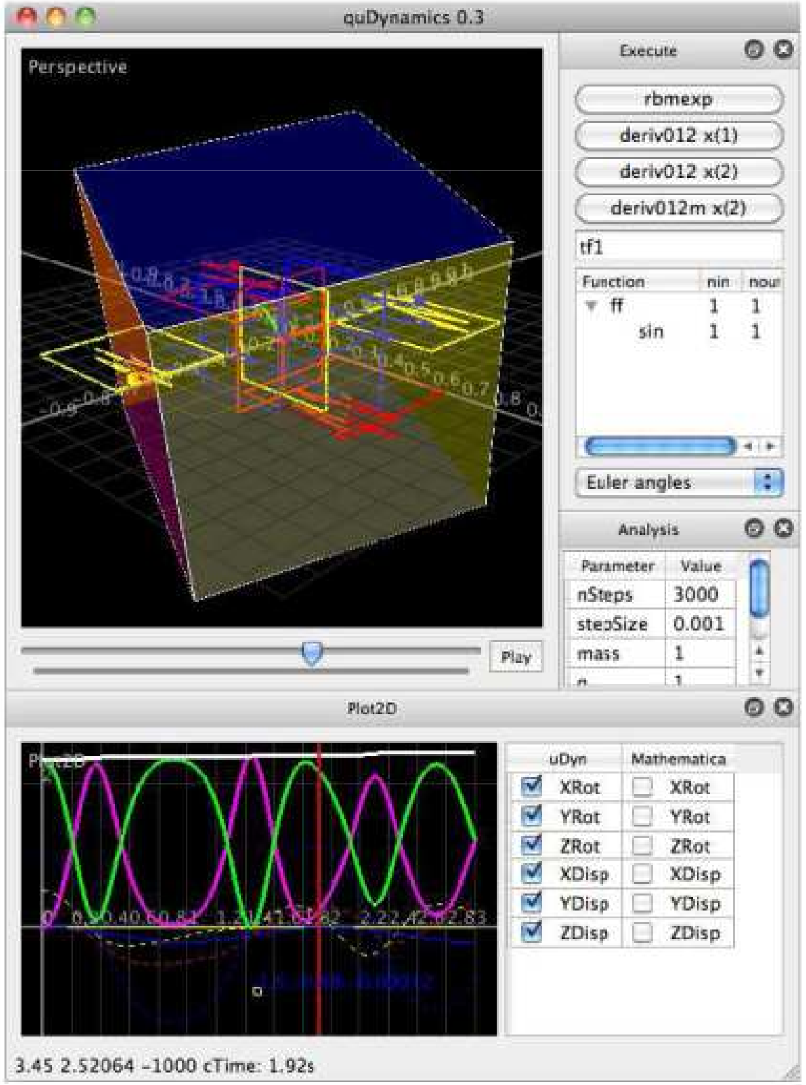
that allow for a **natural conservation of some important quantities** (total energy, momentum etc.), even for considerably **large time-steps**.



Development of a Finite Element code



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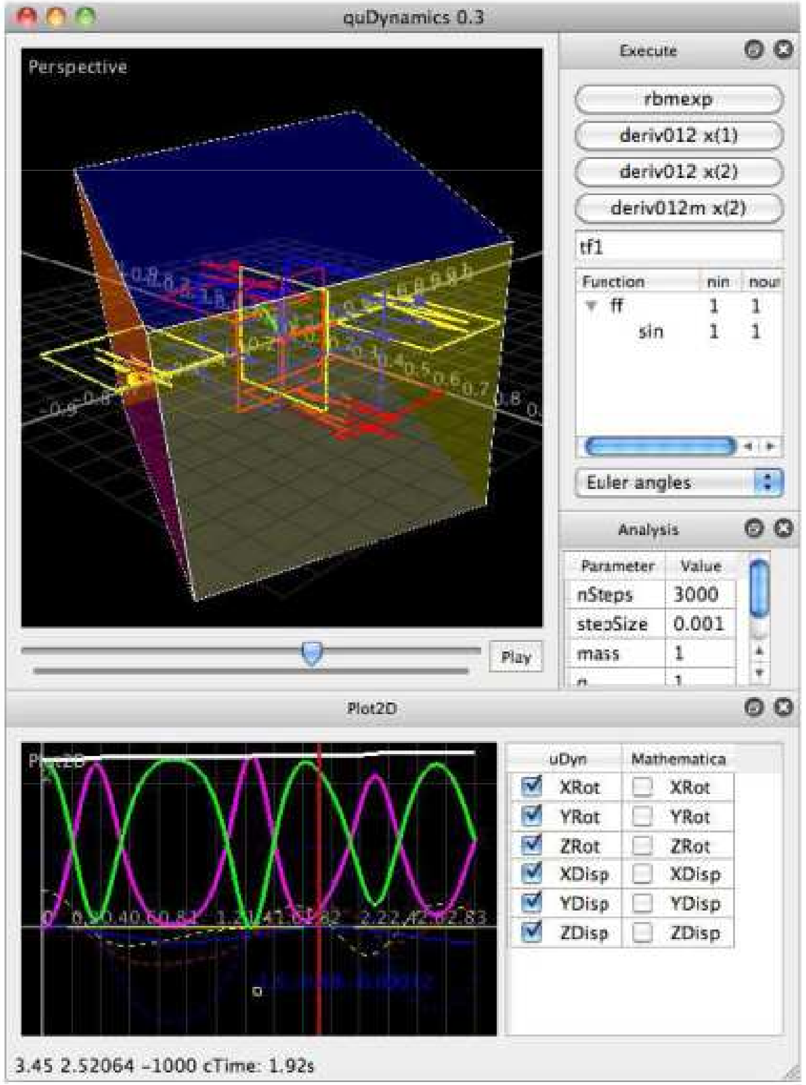

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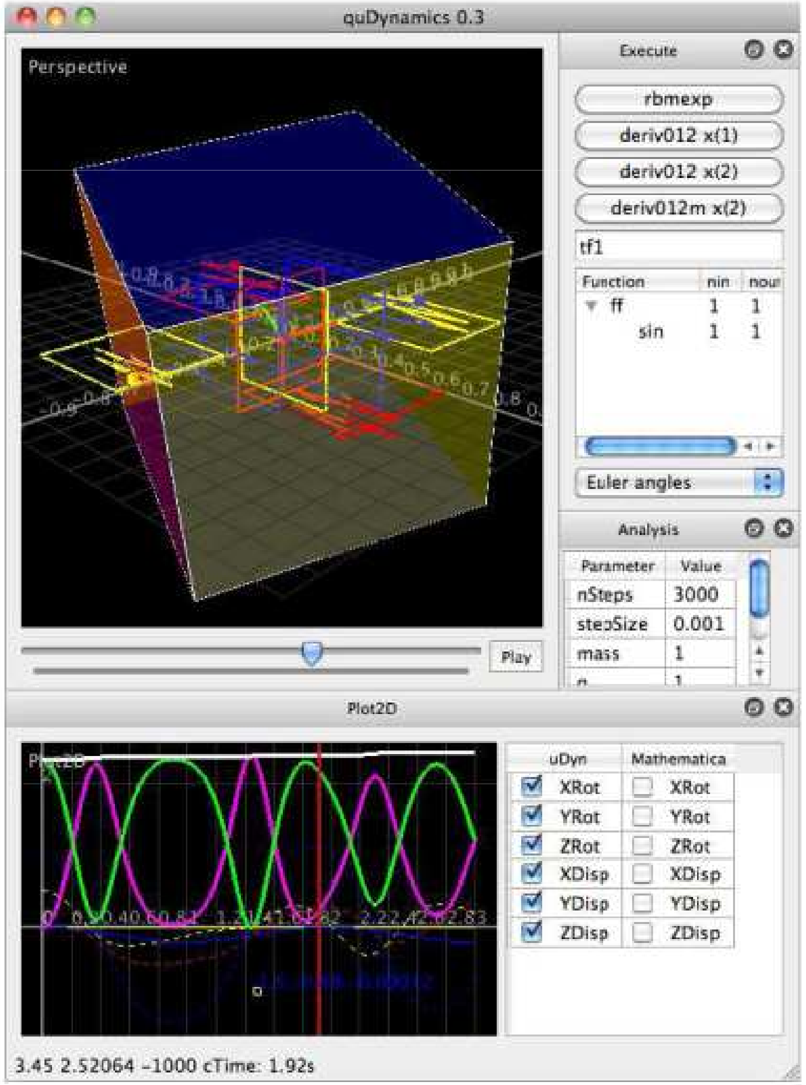

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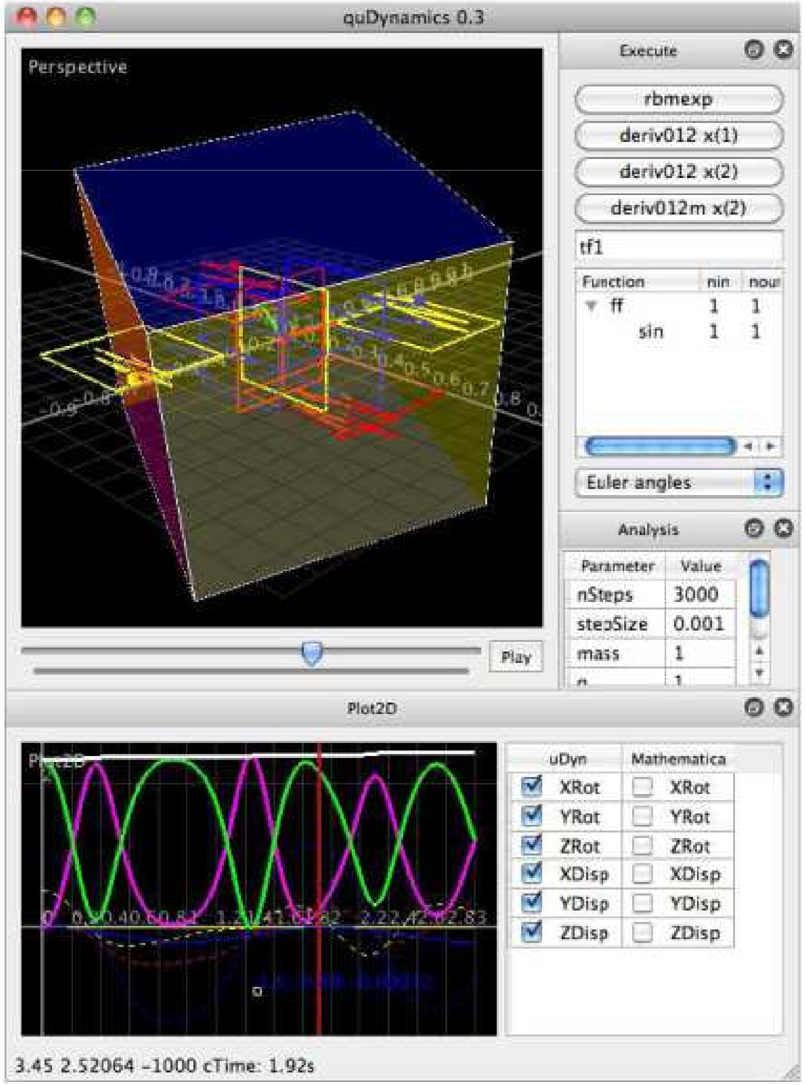

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The program is structured as:

- Computational core implemented in Fortran90.
- *Graphical User Interface* and memory management implemented in C++, using the *cross-platform* framework Qt.

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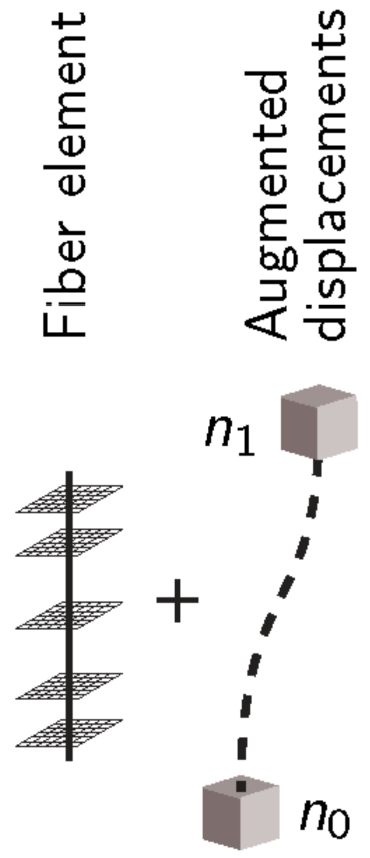

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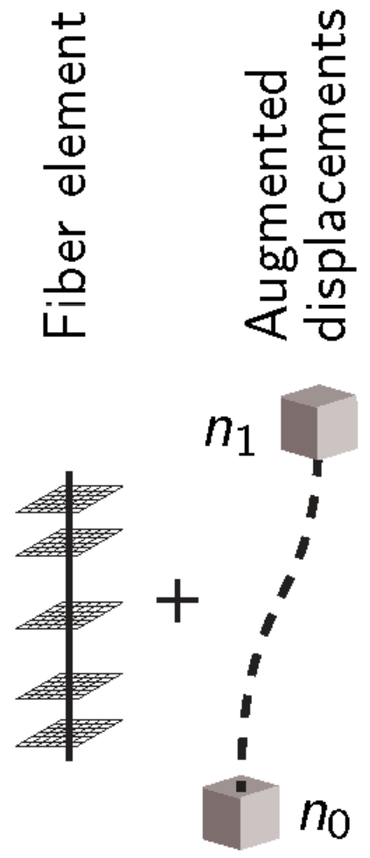

Partial conclusion

From the study and the current stage of development, it emerged the great importance of considering the geometric nature of the structural dynamic problem.

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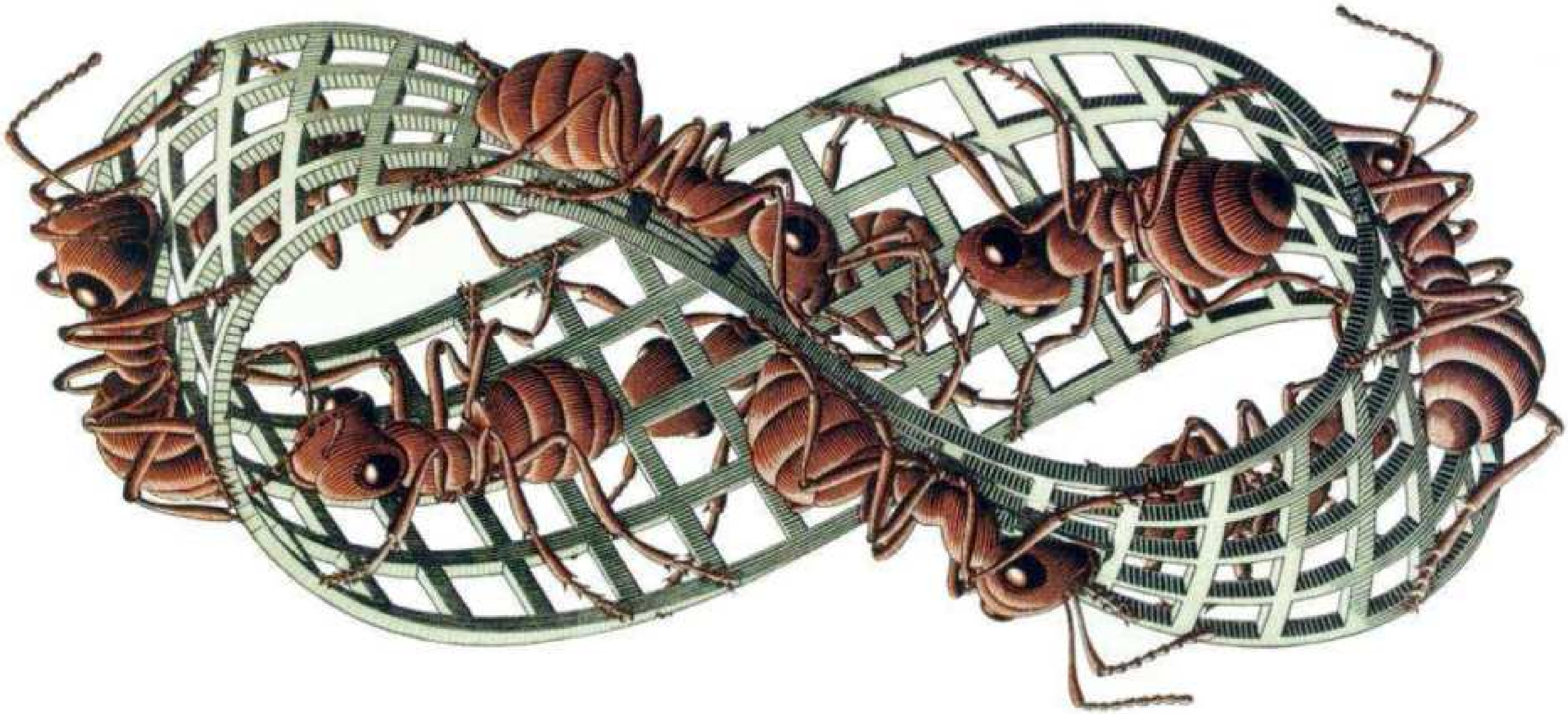
Thank you for your attention!



Maurits Cornelis Escher, *"Moebius strip II"*, 1963