Generic Modelling of Concrete-Steel Composite Beams with Partial Interaction

Gianluca RANZI
Postgraduate Student
University of New South Wales
Sydney, NSW, Australia

Mark BRADFORD
Professor
University of New South Wales
Sydney, NSW, Australia

Brian UY
Associate Professor
University of New South Wales
Sydney, NSW, Australia

Summary

This paper describes a modelling of composite steel-concrete beams with elastic shear connection from a generic standpoint. It builds on the seminal model of Newmark, Siess and Viest, so as to develop the partial interaction formulation for solutions under a variety of end conditions, and lends itself well for modification to quantify directly effects such as shrinkage, creep, and limited slip capacity. This application is possible because the governing differential equations are set up and solved in a fashion whereby inclusion of the kinematic and static end conditions merely requires a statement of the appropriate constants of integration that are generated in the solution of the linear differential equations. The method is applied in the paper for the solution of the well-studied behaviour of simply supported beams with partial interaction. It also provides solutions for a beam encastré at its ends, and for a propped cantilever.

Keywords: Composite beams ; differential equations ; elasticity ; indeterminate ; interface slip ; partial interaction ; shear connection ; T-beams.

1. Introduction

Composite steel-concrete beams are used extensively in contemporary engineering structures, since the attributes that best suit both the concrete (its relatively high compressive strength) and the steel (its high tensile strength) are utilised most favourably. The fundamental characteristic of composite beams that enables composite action to be mobilised is the shear connection between the concrete slab and steel joist, and this paper is concerned with the stiffness aspects of the shear connection, that is partial interaction (PI) at the interface. The essential mechanical shear connection is most often achieved by the use of headed stud connectors [1], whose load-slip characteristics can be determined from standard push-out tests. At service load levels, the three components of the composite beam, viz. the concrete slab, shear connection and the steel joist, can be considered to behave in a fashion that is characterised by linear material response. The elastic response of the shear connection produces PI, and the mechanics of a composite beam with PI makes the analysis of composite members much more difficult than would appear at face value. The influence of the elastic shear connection was addressed over half a century ago in the seminal and highly quoted work of Newmark, Siess and Viest [2], which established that, the solution is far more complex than the familiar midspan deflection of $(5/384)wL^4/EI$ for an elastic beam under a uniformly distributed load.

Many investigators have utilised or extended the work of Newmark for specific applications, eg. a mixed formulation has fairly recently been used in several applications to extend the analysis of PI to investigate the influence of the limited slip capacity of headed stud connectors [3]. This paper formulates an analysis of a steel-concrete composite T-beam with elastic shear connection in a more generic fashion to that usually developed in application-specific treatments. It produces analytical results for beams with a number of support conditions, by solving the linear differential equation that is established in the paper in terms of constants of integration whose prescription for the support conditions is routine by invoking the kinematic and static boundary conditions. The

formulation is demonstrated for the well-known case of a simply supported beam with a uniformly distributed load, and also for a beam that is encastré at both ends and for a propped cantilever beam.

2. Partial Interaction Analysis

The modelling of PI developed here is based on the composite cross-section shown in Fig. 1. For simplicity, a single span beam is considered (Fig. 2), subjected to loading that produces a bending moment $M(z)$ whose variation is not necessarily known initially if the beam is statically indeterminate. Again for simplicity, it is assumed that both the concrete and steel behave elastically in both compression and tension (so that slab cracking is ignored), but modifying the procedure to include this is not difficult. It is assumed further that the shear connection between the concrete slab and steel is elastic, with a modulus k (force per length^2) that defines the relationship between the shear flow force q per unit length and the slip s at the interface by the relationship $q = k \times s$, as shown in Fig. 3.

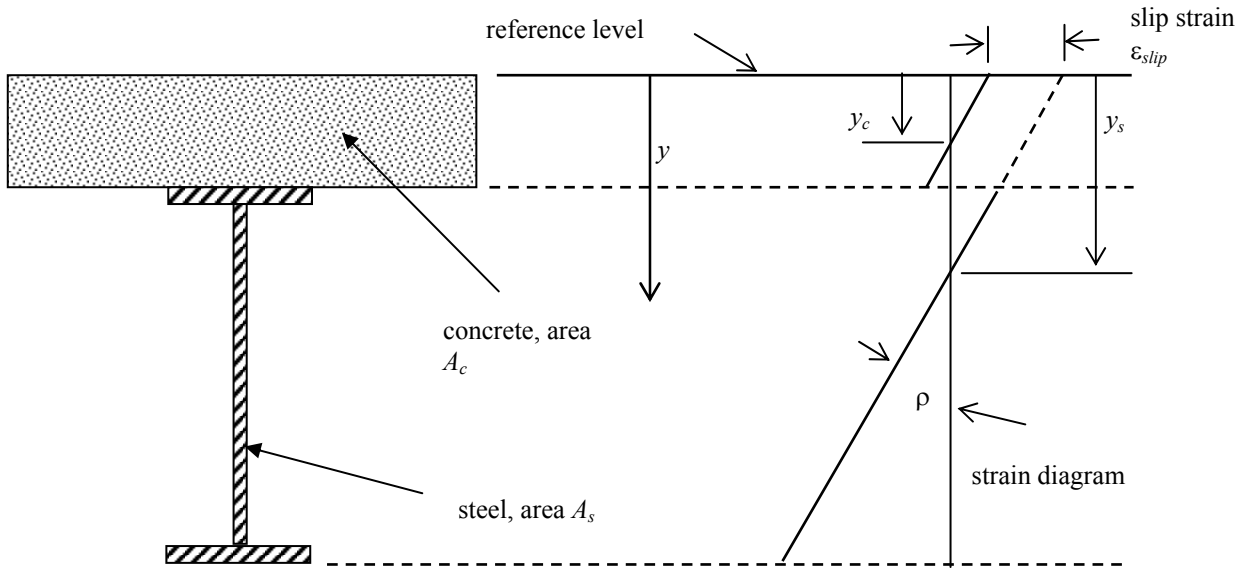


Fig. 1 Composite cross-section

The top fibre of the cross-section is selected as a convenient reference position from which cross-sectional properties are defined, and it is assumed that the curvature ρ is the same in both the concrete and steel, and that plane sections remain plane with a slip discontinuity ds/dz (the slip strain) at the interface. The axial forces in the concrete (N_c) and steel (N_s) are

$$N_c = \int_{A_c} \sigma_c \, \mathrm{d}A_c; \qquad N_s = \int_{A_s} \sigma_s \, \mathrm{d}A_s \tag{1}$$

where A_c and A_s = areas of the concrete and steel respectively, and for each material the stresses are

$$\sigma_c = E_c \varepsilon_c; \qquad \sigma_s = E_s \varepsilon_s \tag{2}$$

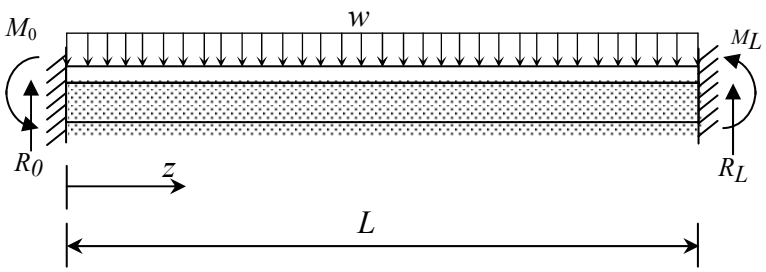


Fig. 2 Generalised single span (redundant) beam

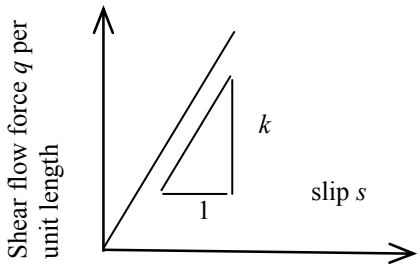


Fig. 3 Response of shear connection

in which E_c and E_s = elastic moduli of the concrete and steel respectively, and the strains are given by

$$\varepsilon_c = (y - y_c)\rho; \qquad \varepsilon_s = (y - y_s)\rho \tag{3}$$

in which y is the distance below the reference position, y_c is the coordinate of the neutral axis for the concrete and y_s is that for the slab, as shown in Fig. 1. From eqns. (1), (2) and (3)

$$N_c = (B_c - y_c A_c)E_c\rho; \qquad N_s = (B_s - y_s A_s)E_s\rho, \tag{4}$$

where B_c and B_s = first moments of area of the concrete and steel respectively below the reference position. Since the beam is subjected to bending only, horizontal equilibrium requires that

$$N_c + N_s = 0 \Rightarrow \qquad \rho[(B_c - y_c A_c)E_c + (B_s - y_s A_s)E_s] = 0, \tag{5}$$

the relationship between the neutral axis depths is given by

$$y_c = \frac{\overline{BE} - y_s A_s E_s}{A_c E_c}; \qquad y_s = \frac{\overline{BE} - y_c A_c E_c}{A_s E_s} \tag{6}$$

where $\overline{BE} = B_c E_c + B_s E_s$. The slip strain ε_{slip} between the steel and concrete is (Fig. 1)

$$\varepsilon_{slip} = \frac{\mathrm{d}s}{\mathrm{d}z} = \varepsilon_c - \varepsilon_s = (y_c - y_s)\rho \Rightarrow \quad y_c = \frac{\overline{BE}}{\overline{AE}} - \frac{A_c E_c}{\overline{AE}\rho}\varepsilon_{slip}; y_s = \frac{\overline{BE}}{\overline{AE}} - \frac{A_c E_c}{\overline{AE}\rho}\varepsilon_{slip} + \frac{\varepsilon_{slip}}{\rho}; \quad \left[\overline{AE} = A_c E_c + A_s E_s\right] \tag{7}$$

The internal bending moment within a cross-section is given by

$$M_{\text{int}} = \int_A \sigma y \,\mathrm{d}A = \int_{A_c} E_c \varepsilon_c y \,\mathrm{d}A_c + \int_{A_s} E_s \varepsilon_s y \,\mathrm{d}A_s \tag{8}$$

Hence, substituting eqn. (3) into eqn. (8) gives

$$M_{\text{int}} = \overline{EI}\rho - (B_c E_c y_c + B_s E_s y_s)\rho; \qquad \overline{EI} = E_c I_c + E_s I_s \tag{9}$$

where I_c, I_s = second moments of area about the reference position. Substituting eqns. (7) into (9) gives

$$M_{\text{int}} = \left(\overline{EI} - \frac{\overline{BE}^2}{\overline{AE}}\right)\rho + \frac{\overline{BE}}{\overline{AE}}\varepsilon_{slip} A_c E_c - B_c E_c \varepsilon_{slip} \tag{10}$$

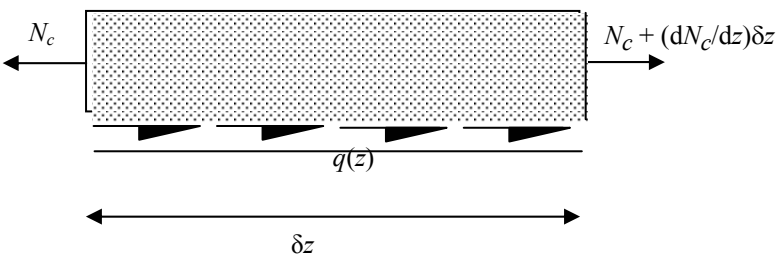


Fig. 4 shows a free body diagram of the slab of length δz with its shear connection, producing

$$\left(N_c + \frac{\mathrm{d}N_c}{\mathrm{d}z}\delta z + q\delta z\right) - N_c = 0 \tag{11}$$

and so

$$\frac{\mathrm{d}N_c}{\mathrm{d}z} = -ks(z) \tag{12}$$

Fig. 4 Free body diagram of slab at interface

If eqn. (7) is substituted into eqn. (4) for the concrete slab, then

$$N_c = B_c E_c \rho - A_c E_c \frac{\overline{BE}}{\overline{AE}}\rho - \frac{\overline{\overline{AE}}}{\overline{AE}}\varepsilon_{slip} \Rightarrow \quad \frac{\mathrm{d}N_c}{\mathrm{d}z} = \left(B_c E_c - \frac{A_c E_c \overline{BE}}{\overline{AE}}\right)\frac{\mathrm{d}\rho}{\mathrm{d}z} - \frac{\overline{\overline{AE}}}{\overline{AE}}\frac{\mathrm{d}^2 s}{\mathrm{d}z^2} \quad \left[\overline{\overline{AE}} = A_c E_c A_s E_s\right] \tag{13}$$

which when using the equilibrium equations (11) and (12) gives

$$\left(B_c E_c - \frac{A_c E_c \overline{BE}}{\overline{AE}}\right)\frac{\mathrm{d}\rho}{\mathrm{d}z} - \frac{\overline{\overline{AE}}}{\overline{AE}}\frac{\mathrm{d}^2 s}{\mathrm{d}z^2} = -ks. \tag{14}$$

The internal moment M_{int} = applied moment $M(z)$ (denoted M for simplicity), so that from eqn. (10),

$$\rho = \gamma M + \alpha \varepsilon_{slip} \tag{15}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}z} = \gamma \frac{\mathrm{d}M}{\mathrm{d}z} + \alpha \frac{\mathrm{d}^2 s}{\mathrm{d}z^2}; \quad \left[\alpha = \frac{B_c E_c \overline{AE} - A_c E_c \overline{BE}}{\overline{AE}\,\overline{EI} - \overline{BE}^2}; \quad \gamma = \frac{\overline{AE}}{\overline{AE}\,\overline{EI} - \overline{BE}^2} \right]. \tag{16}$$

Substituting eqn. (16) into eqn. (14) produces the generic form of the differential equation for PI as

$$\overline{\alpha} \frac{\mathrm{d}^2 s}{\mathrm{d}z^2} - ks = \alpha \frac{\mathrm{d}M}{\mathrm{d}z}; \quad \left[\overline{\alpha} = -\frac{(B_c E_c)^2 A_s E_s + (B_s E_s)^2 A_c E_c - \overline{\overline{AE}}\,\overline{EI}}{\overline{AE}\,\overline{EI} - \overline{BE}^2} \right]. \tag{17}$$

The generic equation (17) may be solved routinely to produce the slip and slip strain respectively as

$$s = C_1 e^{\nu z} + C_2 e^{-\nu z} - \frac{\alpha}{k}\frac{\mathrm{d}M}{\mathrm{d}z} \qquad \varepsilon_{slip} = \nu C_1 e^{\nu z} - \nu C_2 e^{-\nu z} - \frac{\alpha}{k}\frac{\mathrm{d}^2 M}{\mathrm{d}z^2}; \quad \left[\nu^2 = \frac{k}{\overline{\alpha}} \right]. \tag{18a,b}$$

The expression for the curvature can be obtained from eqns. (16) and (18) and integrated to give

$$\rho = \gamma M + \alpha\nu C_1 e^{\nu z} - \alpha\nu C_2 e^{-\nu z} - \frac{\alpha^2}{k}\frac{\mathrm{d}^2 M}{\mathrm{d}z^2}; \quad \theta = \gamma \int M\,\mathrm{d}z + \alpha C_1 e^{\nu z} + \alpha C_2 e^{-\nu z} - \frac{\alpha^2}{k}\int \frac{\mathrm{d}^2 M}{\mathrm{d}z^2}\mathrm{d}z + \hat{C}_1$$

$$v = \gamma \int\int M\,\mathrm{d}z + \frac{\alpha C_1 e^{\nu z}}{\nu} - \frac{\alpha C_2 e^{-\nu z}}{\nu} - \frac{\alpha^2}{k}\int\int \frac{\mathrm{d}^2 M}{\mathrm{d}z^2}\mathrm{d}z + \hat{C}_1 z + \hat{C}_2 \tag{19a,b,c}$$

and the axial force in the concrete is given after appropriate substitution into eqn. (13) as

$$N_c = \gamma_1 \rho + \gamma_2 \varepsilon_{slip}; \quad \left[\gamma_1 = B_c E_c - A_c E_c \frac{\overline{BE}}{\overline{\overline{AE}}}; \quad \gamma_2 = -\frac{\overline{\overline{AE}}}{\overline{\overline{AE}}} \right]. \tag{20}$$

3. Applications

In the redundant beam in Fig. 2, the bending moment M along the beam is defined as

$$M = -M_0 + R_0 z - \frac{wz^2}{2} \tag{21}$$

where R_0 and M_0 are the vertical reaction and the moment at the left support, and w is the uniformly distributed load. The slip and slip strain can then be expressed using eqns. (18a) and (18b) as

$$s = C_1 e^{\nu z} + C_2 e^{-\nu z} - \frac{R_0 - wz}{k}\alpha; \qquad \varepsilon_{slip} = \nu C_1 e^{\nu z} - \nu C_2 e^{-\nu z} + \frac{w}{k}\alpha \tag{22a,b}$$

and so the curvature, rotation and deflection are, respectively

$$\rho = \gamma\left(-M_0 + R_0 z - \frac{wz^2}{2}\right) + \alpha\nu C_1 e^{\nu z} - \alpha\nu C_2 e^{-\nu z} + \frac{\alpha^2}{k} w; \theta = \gamma\left(-M_0 z + \frac{R_0 z^2}{2} - \frac{wz^3}{6}\right) + \alpha C_1 e^{\nu z} + \alpha C_2 e^{-\nu z} + \frac{\alpha^2}{k} wz + \hat{C}_1$$

$$v = \gamma\left(-\frac{M_0 z^2}{2} + \frac{R_0 z^3}{6} - \frac{wz^4}{24}\right) + \frac{\alpha C_1 e^{\nu z}}{\nu} - \frac{\alpha C_2 e^{-\nu z}}{\nu} + \frac{\alpha^2}{k}\frac{wz^2}{2} + \hat{C}_1 z + \hat{C}_2. \tag{23a,b,c}$$

The kinematic boundary conditions $\{ v(z=0) = 0;\ v(z=L) = 0 \} \Rightarrow \{\hat{C}_1;\ \hat{C}_2\}$ may be determined as

$$\hat{C}_1 = \gamma\left(\frac{M_0 L}{2} - \frac{R_0 L^2}{6} + \frac{wL^3}{24}\right) - \frac{\alpha C_1 e^{\nu L}}{\nu L} + \frac{\alpha C_2 e^{-\nu L}}{\nu L} - \frac{\alpha^2}{k}\frac{wL}{2} - \frac{\alpha C_2}{\nu L} + \frac{\alpha C_1}{\nu L}; \quad \hat{C}_2 = \frac{\alpha}{\nu}(C_2 - C_1). \tag{24a,b}$$

The reactions for a simply supported beam can be determined from statics as

$$R_0 = R_L = \frac{wL}{2} \quad , \quad M_0 = M_L = 0 \tag{25}$$

The slip and slip strain are then determined using eqns. (22a) and (22b), and applying the boundary conditions $\{ \varepsilon_{slip}(z=0) = 0;\ \varepsilon_{slip}(z=L) = 0 \} \Rightarrow \{C_1 \text{ and } C_2\}$. Hence these constants of integration are

$$s=\frac{w\psi_1}{\alpha\nu}\left(-e^{\nu(z-L)}+e^{-\nu z}\right)-\frac{w\alpha}{k}\left(\frac{L}{2}-z\right);\quad \varepsilon_{slip}=\frac{w\psi_1}{\alpha}\left(-e^{\nu(z-L)}-e^{-\nu z}\right)+\frac{w\alpha}{k};\quad \psi_1=\frac{\alpha^2}{k\left(e^{-\nu z}+1\right)} \tag{26a,b}$$

The curvature, rotation and deflection can be determined from eqns. (23) and (24) as

$$\rho=w\left[-\frac{\gamma z^2}{2}+\frac{\gamma Lz}{2}+\frac{\alpha^2}{k}-\psi_1\left(e^{-\nu z}+e^{\nu(z-L)}\right)\right]\quad \theta=w\left[-\frac{\gamma z^3}{6}+\frac{\gamma Lz^2}{4}+\frac{\alpha^2 z}{k}-\frac{\gamma L^3}{24}-\frac{L\alpha^2}{2k}-\frac{\psi_1}{\nu}\left(-e^{-\nu z}+e^{\nu(z-L)}\right)\right]$$

$$v=w\left[-\frac{\gamma z^4}{24}+\frac{\gamma Lz^3}{12}+\frac{\alpha^2 z^2}{2k}-\left(\frac{\gamma L^3}{24}+\frac{L\alpha^2}{2k}\right)z+\frac{\alpha^2}{k\nu^2}-\frac{\psi_1}{\nu^2}\left(e^{-\nu z}+e^{\nu(z-L)}\right)\right]. \tag{27a,b,c}$$

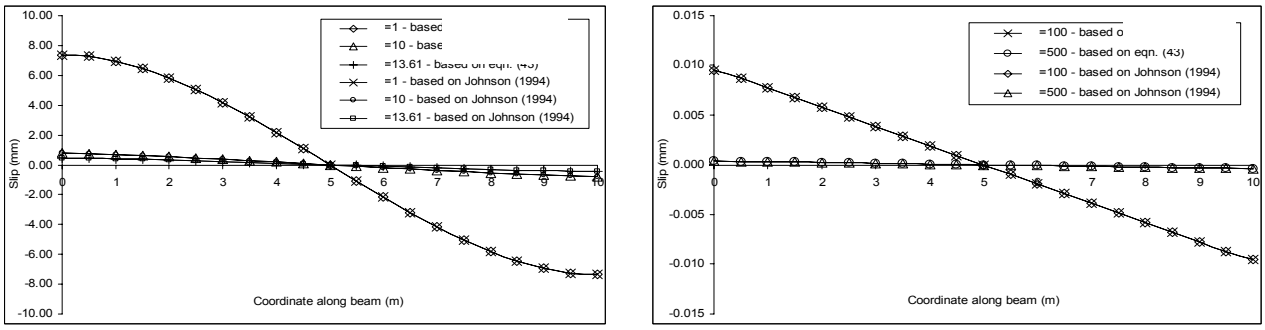


Fig. 5 Slip along 10 m long simply supported beam with a uniformly distributed load

Fig. 5 shows that the results of the slip calculated using eqn. (26) are identical to those presented in Johnson [4]. This behaviour is illustrated for various values of the dimensionless stiffness νL, and in particular for νL = 13.61 that has been worked as a practical example in Ref. [4] with E_1 = 20 kN/mm^2 for the slab 600 wide by 300 deep and E_2 = 200 kN/mm^2 for the joist 60 wide by 300 deep.

For a beam that is encastré at its ends, $R_0 = R_L = wL/2$, $M_0 = -M_L$. The static boundary conditions $s(z=0)=0$; $s(z=L)=0$ lead to the constants of integration being

$$C_1=\frac{1}{2k}\frac{e^{-\nu L}\alpha wL}{e^{-\nu L}-1};\quad C_2=-\frac{1}{2k}\frac{\alpha wL}{e^{-\nu L}-1}, \tag{28a,b}$$

which when substituted into eqns. (26a) and (26b) yield the expressions of the slip and slip strain as

$$s=\frac{wL\psi_2}{\alpha}\left(e^{\nu(z-L)}-e^{-\nu z}\right)-\frac{w\alpha}{k}\left(\frac{L}{2}-z\right);\quad \varepsilon_{slip}=\frac{wL\nu\psi_2}{\alpha}\left(e^{\nu(z-L)}+e^{-\nu z}\right)+\frac{w\alpha}{k};\quad \left[\psi_2=\frac{\alpha^2}{2k\left(e^{-\nu z}-1\right)}\right]. \tag{29a,b}$$

The value of the moment at the supports is calculated imposing $\theta(z=0)=0$, which yields $M_0 = -M_L = wL^2/12$. This implies the points of contraflexure for a beam encastré at both ends are independent of the value of the shear connection stiffness and are located at the same position as those for a beam with full interaction. Using eqns. (28), (29a) and (29b) in eqns. (19a), (19b) and (19c), the distribution of the curvature, slope and deflection for an encastré beam are

$$\rho=w\left[-\frac{\gamma z^2}{2}+\frac{\gamma Lz}{2}-\frac{\alpha L^2}{12}+\frac{\alpha^2}{k}+\psi_2\nu L\left(e^{\nu(z-L)}+e^{-\nu z}\right)\right];\quad \theta=w\left[-\frac{\gamma z^3}{6}+\frac{\gamma Lz^2}{4}+\left(\frac{\alpha^2}{k}-\frac{\alpha L^2}{12}\right)z-\frac{\alpha^2 L}{2k}+\psi_2 L\left(e^{\nu(z-L)}-e^{-\nu z}\right)\right]$$

$$v=w\left[-\frac{\gamma z^4}{24}+\frac{\gamma Lz^3}{12}+\left(\frac{\alpha^2}{k}-\frac{\alpha L^2}{12}\right)\frac{z^2}{2}-\frac{\alpha^2 Lz}{2k}+\frac{\psi_2 L}{\nu}\left(e^{\nu(z-L)}+e^{-\nu z}-e^{-\nu L}-1\right)\right]. \tag{30a,b,c}$$

The left hand end of a propped cantilever ($z = 0$) is assumed to be fixed while the right hand end ($z = L$) is assumed to be a roller support, so that from statics $R_0 = wL - R_L$; $M_L = 0$; $M_0 = wL^2/2 - R_L L$ and so $s(z=0)=0$ and $\varepsilon_s(z=L)=0$ produce

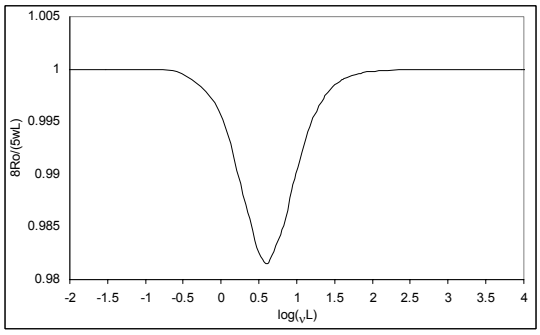
$$C_1=\frac{\alpha e^{-\nu L}}{k\nu}\frac{\nu R_0 e^{-\nu L}-w}{1+e^{-2\nu L}};\qquad C_2=\frac{\alpha}{k\nu}\frac{\nu R_0+we^{-\nu L}}{1+e^{-2\nu L}} \tag{31a,b}$$

$$\rho = -\frac{w\gamma}{2}z^2 + \gamma R_0 z + \frac{w\gamma L^2}{2} - R_0\gamma L + \frac{\alpha^2 w}{k} + \alpha\nu C_1 e^{\nu z} - \alpha\nu C_2 e^{-\nu z};$$

$$\theta = -\frac{w\gamma}{6}z^3 + \frac{\gamma R_0}{2}z^2 + \left(\frac{w\gamma L^2}{2} - R_0\gamma L + \frac{\alpha^2}{k}w\right)z + \alpha C_1 e^{\nu z} + \alpha C_2 e^{-\nu z} + \hat{C}_1$$

$$v = -\frac{w\gamma}{24}z^4 + \frac{\gamma R_0}{6}z^3 + \left(\frac{w\gamma L^2}{2} - R_0\gamma L + \frac{\alpha^2 w}{k}\right)\frac{z^2}{2} + \frac{\alpha C_1 e^{\nu z}}{\nu} - \frac{\alpha C_2 e^{-\nu z}}{\nu} + \hat{C}_1 z + \hat{C}_2;$$

$$\hat{C}_1 = -\frac{5\gamma L^3 w}{24} + \frac{\gamma L^2 R_0}{3} + \frac{\alpha^2}{L\nu^2 k}\left(w + \nu R_0\right) - \frac{\alpha^2 wL}{2k} - \frac{2\alpha^2\left(we^{-\nu L} + \nu R_0\right)}{L\nu^2 k\left(1 + e^{-2\nu L}\right)}$$



$$\hat{C}_2 = \frac{\alpha^2\left(2we^{-\nu L} - \nu R_0 e^{-2\nu L} + \nu R_0\right)}{\nu^2 k\left(1 + e^{-2\nu L}\right)} \tag{32}$$

$$R_0 = \frac{5wL}{8}\cdot\left\{\frac{\left(5\gamma L^4\nu^2 k - 24\alpha^2 + 12\alpha^2 L^2\nu^2\right)\left(e^{-2\nu L}+1\right) + 48\alpha^2 e^{-\nu L}}{5\nu\left(3\alpha^2\nu L^2 + \gamma L^4\nu k\right)\left(e^{-2\nu L}+1\right) + 15\nu\alpha^2 L\left(e^{-2\nu L}-1\right)}\right\} \tag{33}$$

The end reactions are therefore a function of the stiffness of the shear connection. Dimensionless values of the vertical reaction $8R_0/5wL$ are shown in Fig. 7 for different values of the dimensionless stiffness νL.

Fig. 5 Dimensionless reaction versus stiffness for a propped cantilever

4. Conclusions

A generic model for PI between two elastic materials (the concrete slab and the steel joist herein) in a composite flexural member has been derived. The motivation for derivation is to present a formulation that lends itself to simple application to a number of beam support conditions by invoking the relevant static boundary conditions. The kinematic boundary conditions, known *a priori* for the model, are incorporated within the generic derivation.

The model has been utilised to describe the behaviour of a simply supported beam, whose solution is well documented, and that of statically indeterminate structures, viz. an encastré beam and a propped cantilever subjected to uniformly distributed loading. The influence of the shear connection stiffness on the vertical reactions at the support, as well as on the location of the inflexion point, has been determined for a propped cantilever. Owing to the general non-uniformity of a propped cantilever with partial interaction along its length, there is a range of the connection stiffness over which the reactions and inflexion point vary. However, this variation was shown to be only slight.

The generic representation forms the basis for the techniques to investigate such material nonlinear effects as shrinkage, creep, cracking, limit slip capacity of the connectors, and combinations of these.

5. References

[1] OEHLERS, D.J. and BRADFORD, M.A. *Steel and Concrete Composite Structural Members: Fundamental Behaviour*. Pergamon Press, Oxford, 1995.

[2] NEWMARK, N.M., SIESS, C.P. and VIEST, I.M., "Tests and analysis of composite beams with incomplete interaction", *Proc. of Soc. of Exp. Stress Anal.*, Vol 9, No. 1, 1951, pp. 75-92.

[3] OELHERS, D.J. and SVED, G., "Composite beams with limited-slip-capacity shear connectors", *Journal of Structural Engineering, ASCE,* Vol. 121, No. 6, 1995, pp. 932-938.

[4] JOHNSON, R.P., *Composite Structures of Steel and Concrete*, Blackwell, Oxford, 1994.